Sahlqvist theory for Hybrid Logic

Opgedragen aan Dick de Jongh voor zijn 65ste verjaardag

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In his work and life, Dick de Jongh often emphasized the importance of bringing together different disciplines, of viewing problems from different perspectives, and of gaining insight, understanding and joy by employing Iran's official government policy called

dialogue among civilisations.

All these aspects are combined in the work he did in the last phase of his career: the master of logic program at the ILLC. The dialogue between the world —embodied in the highly mixed group of foreign students— and the ILLC has been enormously beneficial for both sides.
We do not exaggerate if we say that the lives of all three authors of this little work have been enormously enriched through this part of Dick's work.

Thanks Dick.

1 Introduction

Hybrid logic comes with a general completeness result: Every extension with pure axioms of the basic hybrid logic with [name] and [bg] rules is complete [2, 3]. A pure axiom is a formula constructed from nominals only, thus not containing arbitrary proposition letters. Pure axioms correspond to first order frame condition and are quite expressive [2]. For instance, $i \to \neg\Diamond i$ defines the class of irreflexive frames.

We can compare this general result with Sahlqvist's theorem for modal logic, a similar general completeness result. Several questions come to mind. Is every Sahlqvist axiom expressible as a pure axiom? No, the Church–Rosser axiom $\Diamond\Box p \to \Box\Diamond p$ is a counterexample [5]. On the other hand, in the presence of the tense modalities, the answer is yes [6]. This gives us two new questions:

1. Is the extension of the basic hybrid logic with a set of Sahlqvist axioms always complete? That is, does Sahlqvist's theorem go through for hybrid logic?

2. Can we combine the two general completeness results? That is, is every extension of the basic hybrid logic with a set Σ of Sahlqvist axioms and a set Π of pure axioms complete with respect to the class of frames defined by Σ and Π together?

This paper answers both questions. We show that every extension of the basic hybrid logic with canonical modal axioms (and hence, Sahlqvist axioms) is complete even without the [name] and [bg] rules, and we give a Sahlqvist formula σ and a pure formula π such that the basic hybrid logic extended with the axioms σ and π is incomplete even in the presence of the [name] and the [bg] rules.

As a corollary, we solve an embarrassing open problem in hybrid logic: whether Beth's definability property holds (cf. [4] for a discussion of this open problem and some partial results). Another corollary of our analysis is that [name] and [bg] are superfluous not only for the basic hybrid system, but also for every Sahlqvist extension. This is a desirable result, since these rules are non-orthodox in the sense that they involve syntactic side-conditions, much like the Gabbay's irreflexivity rule.

The paper is organized as follows. This section briefly recalls hybrid logic. Section 2 shows Sahlqvist's theorem for hybrid logic. In Section 3 we derive interpolation and Beth's property. Section 4 shows that a combination of Sahlqvist and pure axioms is not guaranteed to be complete. We conclude in Section 5.

2 Hybrid logic

What follows is a short textbook-style presentation of hybrid logic, following [2]. Hybrid logic is the result of extending the basic modal language with a second sort of atomic propositions called nominals, and with satisfaction operators. The nominals behave similar to ordinary proposition letters, except that their interpretation in models is restricted to singleton sets. In other words, nominals act as names for worlds. Satisfaction operators allow one to express that a formula holds at the world named by nominal. For example, $@_i p$ expresses that p holds at the world named by the nominal i.

Formally, let PROP be a countably infinite set of proposition letters and NOM a countably infinite set of nominals.[1] The formulas of the basic hybrid logic are given as follows.

$$\phi ::= p \mid i \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi \mid @_i\phi$$

where $p \in$ PROP and $i \in$ NOM. The truth definition for the nominals is the same as for the proposition letters: our models are of the form $M = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame and V a valuation function for the proposition letters and nominals. The truth definition for the nominals is the same as for the proposition letters: $M, w \models i$ iff $w \in V(i)$. The only difference is in the admissible valuations: only valuation functions are allowed that assign to each nominal a singleton set. The interpretation of the satisfaction operators is as could be expected: $M, w \models @_i\phi$ iff $M, v \models \phi$, where $V(i) = \{v\}$.

Next, let us turn to axiomatizations for this language. Let Δ be the set of axioms given by [agree], [back], [introduction], [ref], and [self-dual], for all $i, j \in$ NOM.

$$\begin{array}{rl}
\text{[agree]} & @_j @_i p \to @_i p \\
\text{[back]} & \Diamond @_i p \to @_i p \\
\text{[introduction]} & i \wedge p \to @_i p \\
\text{[ref]} & @_i i \\
\text{[self-dual]} & @_i p \leftrightarrow \neg @_i \neg p.
\end{array}$$

Let $\mathbf{K}_{\mathcal{H}(@)}$ be the smallest set of formulas containing all tautologies, axioms $\Box(p \to q) \to (\Box p \to \Box q)$, $@_i(p \to q) \to (@_i p \to @_i q)$ for $i \in$ NOM, and the axioms in Δ, closed under *modus ponens*, *uniform substitution of formulas for proposition letters and nominals for nominals*, *generalization* (If $\vdash \phi$ then $\vdash \Box\phi$), and *@-generalization* (If $\vdash \phi$ then $\vdash @_i\phi$). Given a set Σ of hybrid formulas, the logic $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is obtained by adding the formulas in Σ to $\mathbf{K}_{\mathcal{H}(@)}$ as extra axioms, and closing under the given rules.

Theorem 2.1 $\mathbf{K}_{\mathcal{H}(@)}$ *is strongly sound and complete for the class of all frames.*

A sketch of the proof of Theorem 2.1 can be found in [2]. Theorem 2.1 also follows from our results in Section 3.

A general completeness result holds for extensions of the basic logic with pure axioms, provided two extra derivation rules are added to the calculus. A pure axiom is an axiom that contains no proposition letters, only nominals.

$$\begin{array}{rl}
\text{[name]} & \text{If } \vdash @_i\phi \text{ and } i \text{ does not occur in } \phi\text{, then } \vdash \phi \\
\text{[bg]} & \text{If } \vdash @_i\Diamond j \to @_j\phi \text{ where } j \neq i \text{ and } j \text{ does not occur in } \phi\text{, then } \vdash @_i\Box\phi
\end{array}$$

[1] The results discussed in this paper and their proofs also apply if NOM is finite.

Several variants of these rules occur in literature, under names such as *Cov* [5] and *Name and Paste* [2]. It the above shape, the rules first appear in [3].

Let $\mathbf{K}^+_{\mathcal{H}(@)}$ be the logic obtained by adding these two inference rules to $\mathbf{K}_{\mathcal{H}(@)}$. Given a set Σ of hybrid formulas, the logic $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ is obtained by adding the formulas in Σ to $\mathbf{K}^+_{\mathcal{H}(@)}$ as extra axioms, and closing under all rules, including the two extra rules.

Theorem 2.2 ([3]) *Let Σ be any set of pure formulas. Then $\mathbf{K}^+_{\mathcal{H}(@)}\Sigma$ is strongly sound and complete for the frame class defined by Σ.*

One question left open by Theorem 2.2 is *when are the rules needed*. While the proof of Theorem 2.2 is based on a Henkin model construction that crucially depends on the presence of the rules, this does not exclude the possibility of completeness without rules. Another question that remains is if every extension of $\mathbf{K}^+_{\mathcal{H}(@)}$ with Sahlqvist axioms is complete. Recall from the introduction that not every Sahlqvist axiom corresponds to a pure axiom.

3 Sahlqvist completeness for hybrid logic

Consider frames of the form $\mathfrak{F} = \langle W, R, (R_i)_{i\in nom}, (S_i)_{i\in \text{NOM}}\rangle$, where each R_i is a binary relation on W and each S_i is a subset of W. Let us call such frames *non-stanfard frames*, to distinguish them from the ordinary frames. Let us say that such a non-standard frame is *nice* if for each $i \in \text{NOM}$, S_i is a singeleton and $\forall xy(R_ixy \leftrightarrow S_iy)$. A non-standard model is a pair $(\mathfrak{F}, V)$ where $\mathfrak{F}$ is a non-standard frame and the valuation V interprets the proposition letters (the interpretation of the nominals is already given by $\mathfrak{F}$).

Viewing the satisfaction operators as modalities and the nominals as modal constants, we can evaluate hybrid formulas on non-standard models: we simply extend the usual satisfaction definition for modal logic with the following clauses:

$$\begin{array}{rll} \mathfrak{M}, w \Vdash i & \text{iff} & w \in S_i, \\ \mathfrak{M}, w \Vdash @_i\phi & \text{iff} & \exists w'(wR_iw' \text{ and } \mathfrak{M}, w' \Vdash \phi). \end{array}$$

By this change in semantics, the formulas in Δ define properties of non-standard frames. For instance, [self-dual] says that the relations R_i are functional. As a matter of fact, each of the axioms, being in Sahlqvist form, is canonical and has a first-order correspondent, given below.

$$\begin{array}{rl} \text{[agree]} & \forall xyz(R_jxy \wedge R_iyz \to R_ixz) \\ \text{[back]} & \forall xyz(Rxy \wedge R_iyz \to R_ixz) \\ \text{[introduction]} & \forall x(S_ix \to R_ixx) \\ \text{[ref]} & \forall x\exists y(R_ixy \wedge S_iy) \\ \text{[self-dual]} & \forall xyz(R_ixy \wedge R_ixz \to y = z). \end{array}$$

It is not hard to see from these first-order correspondents that the following holds.

Lemma 3.1 *A point-generated non-standard frame $\mathfrak{F}$ is* nice *iff $\mathfrak{F} \models \Delta$.*

By canonicity, the following completeness result follows.[2]

Corollary 3.2 *Let Σ be a set of canonical modal formulas. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is strongly sound and complete for the class of nice non-standard frames validating Σ.*

[2] An apparent technical problem: the axiomatization of $\mathbf{K}_{\mathcal{H}(@)}$ includes the following K axiom for the satisfaction operators: $\vdash @_i(p \to q) \to @_ip \to @_iq$. This axiom relies on an interpretation of satisfaction operators as boxes. On the other hand, in the present section, we treat satisfaction operators as diamonds. Hence, strictly speaking, we need the dual K-axiom: $\vdash \overline{@_i}(p \to q) \to \overline{@_i}p \to \overline{@_i}q$. This problem is only apparent, since the latter is derivable from the former in the presence of [self-dual] and [ref].

With each nice non-standard model $\mathfrak{M} = \langle W, R, (R_i)_{i\in\text{NOM}}, (S_i)_{i\in\text{NOM}}, V\rangle$, we can associate a standard hybrid model $\mathfrak{M}^+ = \langle W, R, V \cup \{(i, S_i) \mid i \in \text{NOM}\})$. In fact, this operation on models is bijective, in the sense that for every standard hybrid model $\mathfrak{M}$, there is exactly one nice non-standard model $\mathfrak{N}$ such that $\mathfrak{M} = \mathfrak{N}^+$. A straightforward inductive argument shows that the operation $(\cdot)^+$ preserves local truth of formulas: for all hybrid formulas ϕ, $\mathfrak{M}, w \models \phi$ iff $\mathfrak{M}^+, w \models \phi$. Moreover, if ϕ contains no nominals or satisfaction operators, then ϕ is valid on the underlying frame of $\mathfrak{M}^+$ iff ϕ is valid on the underlying (non-standard) frame of $\mathfrak{M}^+$. Hence, we obtain the following.

Theorem 3.3 *Let Σ be a set of canonical modal formulas not containing nominals or satisfaction operators. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ is strongly sound and complete for the class of frames defined by Σ.*

Corollary 3.4 *Every extension of $\mathbf{K}_{\mathcal{H}(@)}$ with modal Sahlqvist axioms not containing nominals or satisfaction operators is strongly sound and complete for the class of frames defined by the axioms.*

Gargov and Goranko [5] obtain a similar result for the hybrid language with the global modality, via a slightly different route.

4 Interpolation and Beth's property

An open problem in hybrid logic was whether the basic hybrid logic has Beth's definability property [4]. It is known that interpolation fails [1], but Beth's property is a bit weaker. For Beth's property to follow, we need just a restricted version of interpolation: the interpolant may only contain shared proposition letters but it can contain nominals occurring only in the antecedent or in the consequent. As an immediate corollary of Theorem 1 we obtain this form of interpolation and hence Beth's property.

A logic has *interpolation over proposition letters* if whenever $\phi \to \psi$ is provable, there exists an interpolant θ, such that $\phi \to \theta$ and $\theta \to \psi$ are provable, and all proposition letters occuring in θ occur both in ϕ and in ψ.

Marx [7, Corollary B.4.1] showed that every canonical modal logic which is complete with respect to a universal Horn definable class of frames has interpolation over proposition letters. With the exception of [ref], all first-order correspondents of axioms in Δ are universal Horn sentences. The [ref] is itself a formula without proposition letters. By [8], extending a logic that has interpolation over proposition letters with formulas without proposition letters yields again a logic with interpolation over proposition letters. Hence, we obtain the following.

Theorem 4.1 *Let Σ be a set of canonical modal formulas with universal Horn correspondents. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ has interpolation over proposition letters.*

A modal logic is said to have Beth's definability property if every implicit definition can be made explicit. More concretely, let $\Gamma(p)$ be a set of formulas containing the proposition letter p and possibly other proposition letters and nominals. $\Gamma(p)$ defines p if in all models in which both $\Gamma(p)$ and $\Gamma(p')$ are true at every state, also $p \leftrightarrow p'$ is true at every state.[3] In other words, $\Gamma(p)$ defines p if $\Gamma(p) \cup \Gamma(p') \models^{glo} p \leftrightarrow p'$, where $\models^{glo}$ denotes global entailment. Beth's property states that whenever this obtains, there exists a formula θ in which p does not occur, such that $\Gamma(p) \models^{glo} p \leftrightarrow \theta$. Clearly, θ is an explicit definition of p, relative to the theory $\Gamma(p)$.

Beth's property is a completeness theorem for definitions: it states that every semantic definition corresponds to an explicit, syntactic, definition. A standard argument derives the following from Theorem 4.1.

Theorem 4.2 *Let Σ be a set of canonical modal formulas with universal Horn correspondents. Then $\mathbf{K}_{\mathcal{H}(@)}\Sigma$ has Beth's definability property.*

[3] Here, p' is a proposition letter not occuring in Γ, and $\Gamma(p')$ is the result of replacing all occurences of p by p' in $\Gamma(p)$.

5 Combining pure and Sahlqvist axioms

As we mentioned in the introduction, not every Sahlqvist axiom corresponds to a pure axiom. It is natural to ask if completeness obtains when we extend the basic hybrid logic $\mathbf{K}^+_{\mathcal{H}(@)}$ with a combination of pure and canonical axioms. The answer is negative.

Theorem 5.1 *There is a pure axiom π and a Sahlqvist axiom σ such that the hybrid logic $\mathbf{K}^+_{\mathcal{H}(@)}\{\pi, \sigma\}$ is not complete for the frame class defined by $\pi \wedge \sigma$.*

Proof: Consider the following axioms (the first-order frame conditions they define are given as well):

[cr]	$\Diamond\Box p \to \Box\Diamond p$	$\forall xyz(Rxy \wedge Rxz \to \exists u(Ryu \wedge Rzu))$
[nogrid]	$\Diamond(i \wedge \Diamond j) \to \Box(\Diamond j \to i)$	$\forall xyzu(Rxy \wedge Rxz \wedge Ryu \wedge Rzu \to y = z)$
[func]	$\Diamond p \to \Box p$	$\forall xyz(Rxy \wedge Rxz \to y = z)$

[cr] is a Sahlqvist formula and [nogrid] is pure. As can be easily seen from the first-order correspondents, every frame validating [cr] and [nogrid]validates [func]. However, we claim that [func] is not derivable from [cr] and [nogrid](not even using the [name] and [bg] rules). To see this, consider ω^ω, i.e., the countably branching tree of infinite depth. Let $\mathfrak{F}$ be the general frame for this structure in which the admissible sets are exactly the finite and co-finite sets [2].

Obviously, every axiom of hybrid logic is valid on $\mathfrak{F}$. Furthermore, the set of formulas valid on $\mathfrak{F}$ is closed under all derivation rules of hybrid logic, including [name] and [bg] (the latter follows from the fact that every singleton set is admissible). As we are about to show, $\mathfrak{F} \models$ [cr], $\mathfrak{F} \models$ [nogrid] and $\mathfrak{F} \not\models$ [func]. Recall that $\mathbf{K}^+_{\mathcal{H}(@)}\{[\text{cr}], [\text{nogrid}]\}$ is defined as the *smallest* such set of formulas, i.e., the smallest set of formulas containing all axioms of hybrid logic, [cr]and [nogrid], that is closed under all inference rules (including [name] and [bg]). It now clearly follows that [func] $\notin \mathbf{K}^+_{\mathcal{H}(@)}\{[\text{cr}], [\text{nogrid}]\}$.

To show that $\mathfrak{F} \models$ [cr], suppose $\mathfrak{F}, V, w \Vdash \Diamond\Box p$. Since $V(p)$ admissible, it must be either finite or co-finite. Since w satisfies $\Diamond\Box p$, there must be a point with only successors satisfying p. Since every point in ω^ω has infinitely many successors, it follows that $V(p)$ must be infinite, hence co-finite. It follows that every world has a successor satisfying p, and therefore, $\mathfrak{F}, V, w \models \Box\Diamond p$. This establishes the validity of [cr]. That $\mathfrak{F} \models$ [nogrid] and $\mathfrak{F} \not\models$ [func] is clear. $\Box$

6 Conclusion

In hybrid logic we have two general completeness results: Sahlqvist's theorem and the theorem for pure axioms. We showed that they cannot be combined, at least not in the obvious way. The situation is radically different in tense hybrid logic. Here the combination problem is not relevant, as every Sahlqvist axiom is expressible as a pure axiom. It seems that for hybrid logic a similar conclusion holds as for modal logics with the difference operator. In both cases, there is no general completeness theorem like Venema's SD theorem [9] except in the case of tense logics. Venema speculated that in the non-tense case, one can always get completeness by adding suitable axioms, but there is no general recipe indicating which axioms. The axiom that needs to be added in the case of Theorem 5.1 to restore completeness is easy to find. In fact, [func] itself suffices.

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