# De Jongh's characterization of intuitionistic propositional calculus

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**Abstract**

In his PhD thesis [10] Dick de Jongh proved a syntactic characterization of intuitionistic propositional calculus **IPC** in terms of the Kleene slash. In this note we give a proof of this result using the technique of the universal models.

## 1 Introduction

The problem of a syntactic characterization of intuitionistic propositional calculus **IPC** goes back to Lukasiewicz. In [14], Lukasiewicz conjectured that **IPC** is the only intermediate logic[1] having the disjunction property, i.e., if $\vdash \phi \vee \psi$, then $\vdash \phi$ or $\vdash \psi$. However, Kreisel and Putnam [13] disproved this conjecture by constructing a proper extension of intuitionistic logic satisfying the disjunction property. Later, Wronski [17] showed that there are in fact continuum many intermediate logics with the disjunction property. In [12], Kleene defined the Kleene slash $|$ which is a stronger notion than $\vdash$, i.e., for every set of fomulas $\Gamma$, $\Gamma|\phi$ implies $\Gamma \vdash \phi$. He proved that in intuitionistic logic, $\phi|\phi$ iff $\phi$ has the disjunction property and conjectured that this property uniquely characterizes intuitionistic logic among intermediate logics. In his PhD thesis [10], de Jongh confirmed this conjecture, thus providing a pure syntactic characterization of **IPC**. Recently, Iemhoff obtained a different syntactic characterization of **IPC** in terms of admissible rules (see [6] and [7]). In this note we will give a proof of the de Jongh theorem using the technique of the universal models. The proof is the same as in [10] and [11], with the single difference that our proof uses universal and Henkin models, unlike the original one, which used algebraic terminology.

For the basic notions such as intuitionistic Kripke frames (models), generated subframes (submodels), $p$-morphism, bisimulations, etc., the reader is referred to [3] and [2].

The paper is organized as follows. In §2, we recall the structure of the $n$-universal model and its main properties. The de Jongh formulas will be introduced and a connection with Jankov's characteristic formulas will be shown. In §3, we prove de Jongh's characterization theorem.

[1] Recall that an *intermediate logic* $L$ is a set of formulas closed under substitution and modus ponens such that $\mathbf{IPC} \subseteq L \subseteq \mathbf{CPC}$, where **CPC** is the classical propositional calculus.

# 2 $n$-universal models

## 2.1 The structure of $\mathcal{U}(n)$

Fix a propositional language $\mathcal{L}_n$ consisting of finitely many propositional letters $p_1, \dots, p_n$ for $n \in \omega$, the connectives $\wedge$, $\vee$ and $\to$, and the constant $\bot$. Let $\mathfrak{M}$ be an intuitionistic Kripke model. With every point $w$ of $\mathfrak{M}$, we associate a sequence $i_1 \dots i_n$ such that for $k = 1, \dots, n$:

$$i_k = \begin{cases} 1 & \text{if } w \models p_k, \\ 0 & \text{if } w \not\models p_k \end{cases}$$

We call the sequence $i_1 \dots i_n$ associated with $w$ the *color* of $w$ and denote it by $col(w)$.

Colors are ordered according to a relation $\leq$ such that $i_1 \dots i_n \leq i'_1 \dots i'_n$ if for every $k = 1, \dots, n$ we have that $i_k \leq i'_k$. Thus, the set of colors of length $n$ ordered by $\leq$ forms an $n$-element Boolean algebra. We write $i_1 \dots i_n < i'_1 \dots i'_n$ if $i_1 \dots i_n \leq i'_1 \dots i'_n$ and $i_1 \dots i_n \neq i'_1 \dots i'_n$.

For a Kripke frame $\mathfrak{F} = (W, R)$ and $w, v \in W$, we say that a point $w$ is an *immediate successor* of a point $v$ if $w \neq v$, $vRw$, and there does not exist $u \in W$ with $u \neq v$, $u \neq w$, $vRu$ and $uRw$. We say that a set $A$ *totally covers* a point $v$ and write $v \prec A$ if $A$ is the set of all immediate successors of $v$. If $A$ is a singleton set $\{w\}$, then we write $v \prec w$ instead of $v \prec \{w\}$ . We call the relation $\prec$ the *R-immediate successor relation.* $R$ uniquely determines its $R$-immediate successor relation $\prec$ and vice versa if the relation $\prec$ is given, we define $R$ such that $wRv$ iff there is a set $A$ such that $v \in A$ and $w \underbrace{\prec \dots \prec}_{k-times} A$ for some $k \in \omega$. Then it is easy to see that $\prec$ is the $R$-immediate successor relation. Hence, to define a Kripke frame $(W, R)$, it is sufficient to define $(W, \prec)$.

$A \subseteq W$ is an *anti-chain* if $|A| > 1$ and for every $w, v \in A$, $w \neq v$ implies $\neg(wRv)$ and $\neg(vRw)$. A point $w$ of a Kripke frame (Kripke model) is called *maximal* if $wRv$ implies $w = v$ for every $v \in W$. The set of all maximal points of a frame $\mathfrak{F}$ (model $\mathfrak{M}$) we denote by $max(\mathfrak{F})$ ($max(\mathfrak{M})$).

Now we are ready to define the $n$-universal model of **IPC** for every $n \in \omega$. As we mentioned above, to define $\mathcal{U}(n) = (U(n), R, V)$, it is sufficient to define $(U(n), \prec, V)$. The $n$-universal model $\mathcal{U}(n)$ is the smallest Kripke model satisfying the following three conditions:

1. $max(\mathcal{U}(n))$ consists of $2^n$ points of distinct colors.
2. If $w \in U(n)$ and $col(w) \neq \underbrace{0 \dots 0}_{n-times}$, then for every color $i_1 \dots i_n < col(w)$, there exists $v \in U(n)$ such that $v \prec w$ and $col(v) = i_1 \dots i_n$.
3. For every finite anti-chain $A \subset U(n)$ and every color $i_1 \dots i_n$, such that $i_1 \dots i_n \leq col(u)$ for all $u \in A$, there exists $v \in U(n)$ such that $v \prec A$ and $col(v) = i_1 \dots i_n$.

In the remainder of this section, we state the main properties of the $n$-universal model without proof. All the proofs can be found in [3, Sections 8.6 and 8.7], [4], [1], [16] or [15].

If two models $\mathfrak{M}$ and $\mathfrak{M}'$ are bisimilar in the language $\mathcal{L}_n$ we call them $\mathcal{L}_n$-*bisimilar*. Now we state the crucial property of $n$-universal models.

**Theorem 2.1.** *1. For every Kripke model $\mathfrak{M} = (\mathfrak{F}, V)$, there exists a Kripke model $\mathfrak{M}' = (\mathfrak{F}', V')$ such that $\mathfrak{M}'$ is a submodel of $\mathcal{U}(n)$, $\mathfrak{M}$ and $\mathfrak{M}'$ are $\mathcal{L}_n$-bisimilar and $\mathfrak{F}'$ is a p-morphic image of $\mathfrak{F}$.*

*2. For every finite Kripke model $\mathfrak{M}$, there exists $n \leq |\mathfrak{M}|$ such that $\mathfrak{M}$ is a generated submodel of $\mathcal{U}(n)$.*

Theorem 2.1(1) immediately implies the following corollary.

**Corollary 2.2.** *For every formula $\phi$ in the language $\mathcal{L}_n$, we have that*

$$\vdash_{\mathbf{IPC}} \phi \quad \textit{iff} \quad \mathcal{U}(n) \models \phi.$$

We say that a frame $\mathfrak{F} = (W, R)$ is of *depth* $n < \omega$, and write $d(\mathfrak{F}) = n$ if there is a chain of $n$ points in $\mathfrak{F}$ and no other chain in $\mathfrak{F}$ contains more than $n$ points. If for every $n \in \omega$, $\mathfrak{F}$ contains a chain consisting of $n$ points, then $\mathfrak{F}$ is said to be of *infinite depth*. The *depth* of a point $w \in W$ is the depth of the $w$-generated subframe of $\mathfrak{F}$, i.e. the depth of the subframe of $\mathfrak{F}$ based on the set $R(w) = \{v \in W : wRv\}$. The depth of $w$ we denote by $d(w)$.

Let $\mathcal{H}(n) = (H(n), R, V)$ be the *Henkin model* of **IPC** in the language $\mathcal{L}_n$ (for details about the Henkin model see, e.g, [3, §5.1]). Then, the generated sumbodel of $\mathcal{H}(n)$ consisting of all the points of finite depth is (isomorphic to) $\mathcal{U}(n)$. Therefore, $\mathcal{H}(n)$ can be represented as a disjoint union $\mathcal{H}(n) = \mathcal{U}(n) \sqcup \mathcal{T}(n)$, where $\mathcal{T}(n)$ is a submodel of $\mathcal{H}(n)$ consisting of all the points of infinite depth. Moreover, it can be shown that for every point $w$ in $\mathcal{T}(n)$, there exists a point $v \in U(n)$ such that $wRv$. In other words, universal models are "upper parts" of Henkin models.

As we saw in Corollary 2.2, the $n$-universal model of **IPC** carries all the information about the formulas in $n$-variables. Unfortunately, this is not the case for intermediate logics. The analogue of Corollary 2.2 holds for an intermediate logic $L$ iff $L$ has the finite model property. Nevertheless, using the standard Henkin construction one can prove:

**Theorem 2.3.** *For every intermediate logic $L$ and every formula $\phi$ in the language $\mathcal{L}_n$:*

$$\vdash_L \phi \quad \textit{iff} \quad \mathcal{H}_L(n) \models \phi$$

*where $\mathcal{H}_L(n)$ is the Henkin model of $L$ in the language $\mathcal{L}_n$.*

The next technical facts will be used in the proof of Theorem 3.1 below.

**Proposition 2.4.** *For every intermediate logic $L$, $\mathcal{H}_L(n)$ is a generated submodel of $\mathcal{H}(n)$.*

**Proposition 2.5.** *For every $n \in \omega$ and $1 \leq k \leq n$, the submodels of $\mathcal{U}(n)$ with the carrier sets $V(p_k) = \{w \in U(n) : w \models p_k\}$ and $\hat{V}(p_k) = \{w \in H(n) : w \models p_k\}$ are isomorphic to $\mathcal{U}(n-1)$ and $\mathcal{H}(n-1)$, respectively.*

## 2.2 Formulas characterizing point generated subsets

In this section, we will introduce the so-called De Jongh formulas of **IPC** and prove that they define point generated submodels of universal models. We will also show that they do the same job as Jankov's characteristic formulas for **IPC**. For more details on this topic, we refer to [5, §2.5].

Let $w$ be a point in the $n$-universal model (a point of finite depth in the Henkin model). Let $R(w) = \{v \in U(n) : wRv\}$ and $R^{-1}(w) = \{v \in U(n) : vRw\}$. Now we define formulas $\phi_w$ and $\psi_w$ inductively. If $d(w) = 1$ then let

$$\phi_w := \bigwedge\{p_k : w \models p_k\} \wedge \bigwedge\{\neg p_j : w \not\models p_j\} \text{ for each } k, j = 1, \ldots, n$$

and

$$\psi_w = \neg\phi_w.$$

If $d(w) > 1$, then let $\{w_1, \ldots, w_m\}$ be the set of all immediate successors of $w$. Let

$$prop(w) := \{p_k : w \models p_k\}$$

and

$$newprop(w) := \{p_k : w \not\models p_k \text{ and for all } i \text{ such that } 1 \leq i \leq m,\ w_i \models p_k\}.$$

Let

$$\phi_w := \bigwedge prop(w) \wedge \left( (\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i}) \to \bigvee_{i=1}^{m} \phi_{w_i} \right)$$

and

$$\psi_w := \phi_w \to \bigvee_{i=1}^{m} \phi_{w_i}$$

We call $\phi_w$ and $\psi_w$ the *de Jongh formulas*.

**Theorem 2.6.** *For every $w \in U(n)$ ($w \in H(n)$ such that $d(w)$ is finite):*

- *$R(w) = \{v \in U(n) : v \models \phi_w\}$, i.e., $\phi_w$ defines $R(w)$.*
- *$U(n) \setminus R^{-1}(w) = \{v \in U(n) : v \models \psi_w\}$, i.e., $\psi_w$ defines $U(n) \setminus R^{-1}(w)$.*

*Proof.* We will prove the theorem only for universal models. The proof for Henkin models is similar. We prove the theorem by induction on the depth of $w$. If the depth of $w$ is 1, the theorem holds (every element of the maximum of $\mathcal{U}(n)$ is defined by $\phi_w$).

Now suppose the depth of $w$ is greater than 1 and the theorem holds for those points with depth strictly less than $d(w)$. This means that the theorem holds for every $w_i$, i.e., for each $i = 1, \dots, m$, $\phi_{w_i}$ defines $R(w_i)$ and $\psi_{w_i}$ defines $U(n) \setminus R^{-1}(w_i)$.

First note that by the induction hypothesis, $w \not\models \bigvee_{i=1}^{m} \psi_{w_i}$; hence $w \not\models \bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i}$. Therefore, $w \models \phi_w$. On the other hand, if $v \in U(n)$, $wRv$ and $w \neq v$, so again by the induction hypothesis, $v \models \bigvee_{i=1}^{m} \phi_{w_i}$, and thus $v \models \phi_w$. Therefore, every point in $R(w)$ satisfies $\phi_w$.

Now let $v \notin R(w)$. We specify two cases: either there exists a point $u \in U(n)$ such that $vRu$ and $u \notin R(w) \cup \bigcup_{i=1}^{m} R^{-1}(w_i)$ or such a point does not exist. In the first case, by the induction hypothesis, $u \models \bigvee_{i=1}^{m} \psi_{w_i}$ and $u \not\models \bigvee_{i=1}^{m} \phi_{w_i}$. Therefore, $v \not\models \phi_w$. In the latter case, we again specify two cases: there exists $k \in \omega$ such that either $v \underbrace{\prec \dots \prec}_{k-times} w$ or $v \underbrace{\prec \dots \prec}_{k-times} A$ where $A$ is a finite anti-chain containing $w$ and for every $u' \in A$ we have $u' \prec \{w_1, \dots, w_m\}$. In the first case, by the definition of $\mathcal{U}(n)$, $col(v) < col(w)$; hence $v \not\models \bigwedge prop(w)$. In the latter case, there exists a point $u' \in U(n)$ such that $vRu'$ and $u'$ and $w$ are totally covered by the same anti-chain. If $col(u') \not\geq col(w)$, then by the definition of $\leq$ we have that $u' \not\models \bigwedge prop(w)$, which implies that $v \not\models \bigwedge prop(w)$. If $col(u') > col(w)$, then by the definition of $\mathcal{U}(n)$ we have that $u' \models \bigvee newprop(w)$ and $u' \not\models \bigvee_{i=1}^{m} \phi_{w_i}$. Hence $v \not\models \phi_w$. Thus, $\phi_w$ defines $R(w)$.

Finally, we show that $\psi_w$ defines $U(n) \setminus R^{-1}(w)$. For every $v \in U(n)$, $v \not\models \psi_w$ iff there exists $u \in U(n)$ such that $u \models \phi_w$ and $u \not\models \bigvee_{i=1}^{m} \phi_{w_i}$, which holds iff $u \in R(w)$ and $u \notin \bigcup_{i=1}^{m} R(w_i)$, which, in turn, holds iff $u = w$. Hence, $v \not\models \psi_w$ iff $v \in R^{-1}(w)$. This finishes the proof of the theorem. □

## 2.3 The Jankov formulas

In this subsection we show that the de Jongh formulas do the same job as Jankov's characteristic formulas for **IPC**. We first recall the theorem of Jankov.

**Theorem 2.7.** *(see Jankov [8] and [9]) For every finite rooted frame $\mathfrak{F}$ there exists a formula $\chi(\mathfrak{F})$ such that for every frame $\mathfrak{G}$*

$$\mathfrak{G} \not\models \chi(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a generated subframe of a p-morphic image of } \mathfrak{G}.$$

Now we will give an alternative proof of this theorem using the de Jongh formulas.

**Proof of Theorem 2.7**:

By Theorem 2.1(2) there exists $n \in \omega$ such that $\mathfrak{F}$ is (isomorphic to) a generated subframe of $\mathcal{U}(n)$. Let $w \in U(n)$ be the root of $\mathfrak{F}$. We show that for every frame $\mathfrak{G}$:

$$\mathfrak{G} \not\models \psi_w \text{ iff } \mathfrak{F} \text{ is a generated subframe of a } p\text{-morphic image of } \mathfrak{G}.$$

Suppose $\mathfrak{F}$ is a generated subframe of a $p$-morphic image of $\mathfrak{G}$. Clearly, $w \not\models \psi_w$; hence $\mathfrak{F} \not\models \psi_w$, and since $p$-morphisms preserve the validity of formulas $\mathfrak{G} \not\models \psi_w$.

Now suppose $\mathfrak{G} \not\models \psi_w$. Then, there exists a model $\mathfrak{M} = (\mathfrak{G}, V)$ such that $\mathfrak{M} \not\models \psi_w$. By Theorem 2.1(1), there exists a submodel $\mathfrak{M}' = (\mathfrak{G}', V')$ of $\mathcal{U}(n)$ such that $\mathfrak{G}'$ is a $p$-morphic image of $\mathfrak{G}$ and $\mathfrak{M}$ and $\mathfrak{M}'$ are $\mathcal{L}_n$-bisimilar. This means that $\mathfrak{M}' \not\models \psi_w$. Now, $\mathfrak{M}' \not\models \psi_w$ iff there exists $v$ in $\mathfrak{G}'$ such that $vRw$, which holds iff $w$ belongs to $\mathfrak{G}'$. Therefore, $w$ is in $\mathfrak{G}'$, and $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}'$. Thus, $\mathfrak{F}$ is a generated subframe of a $p$-morphic image of $\mathfrak{G}$.

**Remark 2.8.** Note that there is one essential difference between the Jankov and de Jongh formulas: the number of propositional variables used in the Jankov formula depends on (is equal to) the cardinality of $\mathfrak{F}$, whereas the number of variables in the de Jongh formula depends on which $\mathcal{U}(n)$ contains $\mathfrak{F}$ as a generated submodel. Therefore, in general, the de Jongh formula contains fewer variables than the Jankov formula.

# 3 Characterization of IPC

## 3.1 Characterization using the de Jongh formulas

In this subsection, we prove a characterization of **IPC** using the de Jongh formulas.

Let $L$ be an intermediate logic. A formula $\phi$ is said to have the *$L$-disjunction property* if for all $\psi$ and $\chi$, if $\vdash_L \phi \to \psi \vee \chi$, then $\vdash_L \phi \to \psi$ or $\vdash_L \phi \to \chi$.

**Theorem 3.1.** *Let $L$ be an intermediate logic. $L =$ **IPC** iff for all $m \in \omega$, $\phi_1, \ldots, \phi_m$, and $\psi$*

$$\textit{if } \nvdash_L (\bigvee_{j=1}^{m} \phi_j \to \psi) \to \phi_i \textit{ for every } i \leq m,$$

$$\textit{then } \bigvee_{j=1}^{m} \phi_j \to \psi \textit{ has the } L\textit{-disjunction property.}$$

*Proof.* $\Rightarrow$ Suppose $L = \mathbf{IPC}$ and there are $m \in \omega$, $\phi_1, \ldots, \phi_m$ and $\psi$ falsifying the right hand side of the bi-conditional statement of the theorem. Let $\phi := \bigvee_{j=1}^{m} \phi_j \to \psi$. Then for every $i \leq m$ $\nvdash_{\mathbf{IPC}} \phi \to \phi_i$, and there exist $\chi_1$ and $\chi_2$ such that $\vdash_{\mathbf{IPC}} \phi \to \chi_1 \vee \chi_2$, but $\nvdash_{\mathbf{IPC}} \phi \to \chi_1$ and $\nvdash_{\mathbf{IPC}} \phi \to \chi_2$. Therefore, there exist finite rooted Kripke models $\mathfrak{M}_1, \ldots, \mathfrak{M}_m$ and $\mathfrak{M}'_1$, $\mathfrak{M}'_2$ such that $w_i \models \phi$ and $w_i \not\models \phi_i$, where $w_i$ is the root of $\mathfrak{M}_i$, and $v_k \models \phi$ and $v_k \not\models \chi_k$ for $k = 1, 2$, where $v_1$ and $v_2$ are the roots of $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$, respectively. Take the disjoint union of the models $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$, $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$ and add a new root $w$ to it. Extend the valuation to $w$ (this can be done by letting $w \not\models p_k$ for every proposition letter $p_k$). Then, $w \not\models \bigvee_{j=1}^{m} \phi_j$; hence, $w \models \phi$. On the other hand, $w \not\models \chi_1 \vee \chi_2$, therefore $w \not\models \phi \to \chi_1 \vee \chi_2$, which is a contradiction.

$\Leftarrow$ Suppose $L \supset \mathbf{IPC}$. Since **IPC** is complete with respect to finite rooted Kripke frames, there exists a finite rooted Kripke frame $\mathfrak{F}$ such that $\mathfrak{F}$ is not an $L$-frame. Without loss of generality, we can assume that every proper generated subframe of $\mathfrak{F}$ is an $L$-frame. Let $n$ be the smallest natural number such that $\mathfrak{F}$ is (isomorphic to) a generated subframe of $\mathcal{U}(n)$. Let $w \in U(n)$ be the root of $\mathfrak{F}$ and $\{w_1, \ldots, w_m\}$ be the set of all immediate successors of $w$. We can assume that $m > 1$; otherwise, instead of $\mathfrak{F}$ we take the disjoint union of $\mathfrak{F}$ minus $w$ with itself and add a new root to it.

First, we show that in this case the de Jongh formula $\phi_w$ can be simplified. As we mentioned above, we can assume that $w$ has at least two successors. If there exists a propositional letter $p_k$ such that for every $i \leq m$ we have $w_i \models p_k$, then every $w_i$ belongs to the set $V(p_k) = \{u \in U(n) : u \models p_k\}$. By Proposition 2.5 the submodel of $\mathcal{U}(n)$ with the carrier set $V(p_k)$ is isomorphic to $\mathcal{U}(n-1)$. Therefore, $\bigcup_{i=1}^{m} R(w_i)$ is (isomorphic to) a generated subframe of $\mathcal{U}(n-1)$. Then, there exists a point $v \in U(n-1)$ totally covered by $\{w_1, \ldots, w_m\}$. This implies that $R(v)$ is isomorphic to $\mathfrak{F}$. This is a contradiction since $n$ is the least natural number such that $\mathfrak{F}$ is a generated subframe of $\mathcal{U}(n)$. Therefore, for every proposition letter $p_k$, there is an immediate successor of $w$ falsifying $p_k$. This implies that $newprop(w) = \emptyset$. On the other hand, if there is a $p_k$ such that $w \models p_k$, then every immediate successor of $w$ also satisfies $p_k$, which is a contradiction. Hence, $prop(w) = \emptyset$. Thus, the formula $\phi_w$ is equal to the formula $\bigvee_{i=1}^{m} \psi_{w_i} \to \bigvee_{i=1}^{m} \phi_{w_i}$.

Let $\phi_i := \psi_{w_i}$ for $i \leq m$ and $\psi := \bigvee_{j=1}^{m} \phi_{w_j}$. Now we show that the de Jongh formula $\phi_w = \bigvee_{j=1}^{m} \phi_j \to \psi$ does not have the $L$-disjunction property, even though, for each $i \leq m$ we have $\nvdash_L \phi_w \to \phi_i$. Let $\chi_1 := \bigvee_{j=1}^{m-1} \phi_{w_j}$ and $\chi_2 := \phi_{w_m}$.

The following three claims will finish the proof of the theorem:

1. $\nvdash_L \phi_w \to \psi_{w_i}$ for each $i \leq m$.
2. $\vdash_L \phi_w \to \chi_1 \vee \chi_2$, i.e., $\vdash_L \psi_w$.
3. $\nvdash_L \phi_w \to \chi_1$ and $\nvdash_L \phi_w \to \chi_2$.

(1) By the definition of $\phi_w$ and $\psi_w$, for each $i \leq m$, we have that $w_i \models \phi_w$ and $w_i \not\models \psi_{w_i}$ . Every proper generated subframe of $\mathfrak{F}$ is an $L$-frame; thus, $\nvdash_L \phi_w \to \psi_{w_i}$ for each $i \leq m$.

(2) By Theorem 2.3, it is sufficient to show that $\mathcal{H}_L(n) \models \psi_w$. By Proposition 2.4, we know that $\mathcal{H}_L(n)$ is a generated subframe of $\mathcal{H}(n)$. It is well known that if a finite model $\mathfrak{M} = (\mathfrak{F}, V)$ is a generated submodel of $\mathcal{H}_L(n)$, then $\mathfrak{F}$ is an $L$-frame (see, e.g., [2, Lemma 3.27]). Suppose there is a point $v \in H_L(n)$ such that $v \not\models \psi_w$. Then, by Theorem 2.6, we have $vRw$. But then the submodel of $\mathcal{H}(n)$ with the carrier set $R(w)$ is a finite generated submodel of $\mathcal{H}_L(n)$. This implies that $\mathfrak{F}$ (which is isomorphic to the subframe of $\mathcal{H}(n)$ with the carrier set $R(w)$) is an $L$-frame. This is a contradiction, since $\mathfrak{F}$ is not an $L$-frame.

(3) As we mentioned above, $w_i \models \phi_w$ for each $i \leq m$. On the other hand, $w_1 \not\models \phi_{w_m}$ and $w_m \not\models \phi_{w_i}$ for each $i \leq m-1$. Therefore, $\nvdash_L \phi_w \to \chi_1$ and $\nvdash_L \phi_w \to \chi_2$. □

## 3.2 The Characterization of IPC by the Kleene slash

In this subsection, we prove the de Jongh characterization of **IPC**. Let $L$ be an intermediate logic. Recall from [12] the defintion of the Kleene slash.

**Definition 3.2.** *For every intermediate logic $L$ and a set of formulas $\Gamma$ we say that*

- $\Gamma \mid_L \bot$ *if* $\bot \in \Gamma$.
- $\Gamma \mid_L p$ *if* $p \in \Gamma$ *for every proposition letter* $p$.
- $\Gamma \mid_L \phi \wedge \psi$ *if* $\Gamma \mid_L \phi$ *and* $\Gamma \mid_L \psi$.
- $\Gamma \mid_L \phi \vee \psi$ *if* $\Gamma \mid_L \phi$ *or* $\Gamma \mid_L \psi$.
- $\Gamma \mid_L \phi \to \psi$ *if* ($\Gamma \mid_L \phi$ *implies* $\Gamma \mid_L \psi$) *and* $\Gamma \vdash_L \phi \to \psi$.[2]

**Theorem 3.3.** *For every intermediate logic $L$, every set of formulas $\Gamma$ and every formula $\phi$, we have $\Gamma \mid_L \phi$ implies $\Gamma \ \vdash_L \ \phi$.*

*Proof.* A straightforward induction on the complexity of $\phi$. □

If $\Gamma$ consists of formulas with the $L$-disjunction property, then the converse of Theorem 3.3 also holds. We formulate this result only for the case in which $\Gamma$ is a singleton set.

**Theorem 3.4.** *For every intermediate logic $L$ and formulas $\phi$ and $\psi$. If $\phi$ has the $L$-disjunction property, then*

1. $\phi \mid_L \psi$ *iff* $\phi \vdash_L \psi$.
2. $\phi \mid_L \phi$.

*Proof.* (1) easy induction on the complexity of $\psi$.
(2) follows from (1). □

Now we show that the converse to Theorem 3.4(2), i.e., if $\phi \mid_L \phi$, then $\phi$ has the $L$-disjunction property, holds only in the case of **IPC**.

**Theorem 3.5.** *For every set of formulas $\Gamma$ such that for every $\psi \in \Gamma$ we have $\Gamma \mid_{\mathbf{IPC}} \psi$, it is the case that for every formula $\phi$, $\Gamma \vdash_{\mathbf{IPC}} \phi$ implies $\Gamma \mid_{\mathbf{IPC}} \phi$.*

*Proof.* The theorem is proved by a straightforward induction on the depth of the proof of $\phi$. First, one has to choose his/her favorite proof system of **IPC** and then check that the theorem holds for every axiom of **IPC** and that the required property is preserved by the rules of inference. □

**Theorem 3.6.** *Let $L$ be an intermediate logic. Then $L =$ **IPC** iff*

*for every formula $\phi$, $\phi \mid_L \phi$ iff $\phi$ has the $L$-disjunction property.*

[2]The condition $\Gamma \vdash_L \phi \to \psi$ was added by Aczel, in order to prove Theorem 3.3 (de Jongh, P.C.)

*Proof.* Suppose $L = \mathbf{IPC}$. By Theorem 3.5 $\phi \mid_{\mathbf{IPC}} \phi$ and $\phi \vdash_{\mathbf{IPC}} \psi \vee \chi$ together imply that $\phi \mid_{\mathbf{IPC}} \psi \vee \chi$. By the definition of the Kleene slash, this implies that $\phi \mid_{\mathbf{IPC}} \psi$ or $\phi \mid_{\mathbf{IPC}} \chi$. By Theorem 3.3 we then have that $\phi \vdash_{\mathbf{IPC}} \psi$ or $\phi \vdash_{\mathbf{IPC}} \psi$.

Now suppose $L \supset \mathbf{IPC}$. Then take the formula $\phi_w = \bigvee_{j=1}^{m} \phi_j \to \psi$ constructed in the proof of Theorem 3.1. As was shown in the proof of Theorem 3.1, $\phi_w$ does not have the $L$-disjunction property. Now we will show that $\phi_w \mid_L \phi_w$. Since $\phi_w = \bigvee_{j=1}^{m} \phi_j \to \psi$, for showing $\phi_w \mid_L \phi_w$, we need to prove that $\phi_w \vdash_L \bigvee_{j=1}^{m} \phi_j \to \psi$ and that if $\phi_w \mid_L \bigvee_{j=1}^{m} \phi_j$, then $\phi_w \mid_L \psi$. It is obvious that $\phi_w \vdash_L \phi_w$. Thus, $\phi_w \vdash_L \bigvee_{i=1}^{m} \phi_i \to \psi$. In the proof of Theorem 3.1 we showed that for each $i \leq m$, $\phi_w \nvdash \phi_i$. This implies that $\phi_w \not|_L \bigvee_{i=j}^{m} \phi_j$. Hence, if $\phi_w \mid_L \bigvee_{j=1}^{m} \phi_j$, then $\phi_w \mid_L \psi$. This, by the definition of the Kleene slash, yields $\phi_w \mid_L \phi_w$. Therefore, we have found a formula $\phi_w$ such that $\phi_w \mid_L \phi_w$ and $\phi_w$ does not have the $L$-disjunction property.

□

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