Definability in components

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A downwards linear order is well-founded if and only if all its components are. In his study of definability [D], Doets ran into the question whether a similar invariance holds for *definable* well-foundedness. This question — the direction from right to left is the harder part — is settled below, in some additional generality. Moreover, all the difficult words of this introduction are explained there.

1. A definability theorem

For any set X, let X^* be the set of finite sequences of elements of X.

Let **A** be a structure, fixed for this section, with universe A, for a first order language $\mathcal{L}$. Let us assume for the sake of simplicity that all symbols of $\mathcal{L}$ are relation symbols. (We shall reconsider this assumption below.) Let B be a *component* of **A**: a subset of A with the property that for every symbol R of $\mathcal{L}$, $R^{\mathbf{A}}$, the relation over **A** that is the interpretation of R, is contained in $B^* \cup (A - B)^*$. I write **B** to refer to the substructure of **A** with universe B. I shall call the substructure a component as well; there is no need to require that it cannot be subdivided further.

Theorem 1. Let $\varphi = \varphi(u_1,\ldots, u_k, v_1,\ldots, v_l)$, $\varphi(\boldsymbol{u}, \boldsymbol{v})$ for short, be a formula of $\mathcal{L}$, in the free variables $u_1,\ldots, u_k, v_1,\ldots, v_l$. There exists a function f from $(A - B)^k$ to formulas of $\mathcal{L}$ in $v_1,\ldots, v_l$, with finite range, such that for all $\boldsymbol{a} \in (A - B)^k$,

$$\text{for all } \boldsymbol{b} \in B^l\text{: } \mathbf{A} \models \varphi[\boldsymbol{a}, \boldsymbol{b}] \text{ if and only if } \mathbf{B} \models f(\boldsymbol{a})[\boldsymbol{b}].$$

Proof. By induction on φ. Instead of $f(\boldsymbol{a})$, where f is the function of the theorem for φ and the variables $\boldsymbol{u}$, I shall write $\varphi^{\boldsymbol{a}}_{\boldsymbol{u}}$.

Suppose $\varphi = R\boldsymbol{w}$. If $\boldsymbol{w}$ consists entirely of variables from $\boldsymbol{u}$, we distinguish two cases: if $\mathbf{A} \models \varphi[\boldsymbol{a}]$, we put $\varphi^{\boldsymbol{a}}_{\boldsymbol{u}} = \top$; if $\mathbf{A} \models \neg\varphi[\boldsymbol{a}]$, $\varphi^{\boldsymbol{a}}_{\boldsymbol{u}} = \bot$. If $\boldsymbol{w}$ contains variables from both $\boldsymbol{u}$ and $\boldsymbol{v}$, we may take $\varphi^{\boldsymbol{a}}_{\boldsymbol{u}} = \bot$, because B is a component. Finally, if $\boldsymbol{w}$ consists of variables from $\boldsymbol{v}$, we take $\varphi^{\boldsymbol{a}}_{\boldsymbol{u}} = \varphi$.

The induction step for negation is trivial.

If φ is $\psi \vee \chi$, take $\varphi^{\boldsymbol{a}}_{\boldsymbol{u}} = \psi^{\boldsymbol{a}}_{\boldsymbol{u}} \vee \chi^{\boldsymbol{a}}_{\boldsymbol{u}}$. Since there are finitely many distinct $\psi^{\boldsymbol{a}}_{\boldsymbol{u}}$ and $\chi^{\boldsymbol{a}}_{\boldsymbol{u}}$, there will be finitely many $\varphi^{\boldsymbol{a}}_{\boldsymbol{u}}$.

Suppose $\varphi = \forall x\, \psi(x, \boldsymbol{u}, \boldsymbol{v})$. By induction hypothesis, we have a finite number of formulas $\psi^{\boldsymbol{a}}_{\boldsymbol{u}}(x, \boldsymbol{v})$ and $\psi^{a,\boldsymbol{a}}_{x,\boldsymbol{u}}(\boldsymbol{v})$ such that for all $\boldsymbol{a} \in (A - B)^k$, $b \in B$, and $\boldsymbol{b} \in B^l$:

$$\mathbf{A} \models \psi[b, \boldsymbol{a}, \boldsymbol{b}] \text{ if and only if } \mathbf{B} \models \psi^{\boldsymbol{a}}_{\boldsymbol{u}}[b, \boldsymbol{b}];$$

for all $a \in A - B$, $\boldsymbol{a} \in (A - B)^k$, and $\boldsymbol{b} \in B^l$:

$$\mathbf{A} \models \psi[a, \boldsymbol{a}, \boldsymbol{b}] \text{ if and only if } \mathbf{B} \models \psi^{a,\boldsymbol{a}}_{x,\boldsymbol{u}}[\boldsymbol{b}].$$

Take $\varphi^{\boldsymbol{a}}_{\boldsymbol{u}}$ =

$$\forall x\ \psi^{\boldsymbol{a}}_{\boldsymbol{u}}(x, \boldsymbol{v}) \wedge \bigwedge_{a \in A-B} \psi^{a,\boldsymbol{a}}_{x,\boldsymbol{u}}(\boldsymbol{v}).$$

It is easy to see that this gives us a finite number of (finite) formulas. Moreover, for arbitrary $\boldsymbol{a} \in (A - B)^k$ we have, for any sequence $\boldsymbol{b} \in B^l$:

$\mathbf{A} \models \varphi[\boldsymbol{a}, \boldsymbol{b}]$ if and only if

for all $b \in B$, $\mathbf{A} \models \psi[b, \boldsymbol{a}, \boldsymbol{b}]$, and for all $a \in A - B$, $\mathbf{A} \models \psi[a, \boldsymbol{a}, \boldsymbol{b}]$,

if and only if for all $b \in B$, $\mathbf{B} \models \psi^{\boldsymbol{a}}_{\boldsymbol{u}}[b, \boldsymbol{b}]$, and for all $a \in A - B$, $\mathbf{B} \models \psi^{a,\boldsymbol{a}}_{x,\boldsymbol{u}}[\boldsymbol{b}]$, by induction hypothesis,

if and only if $\mathbf{B} \models \forall x\ \psi^{\boldsymbol{a}}_{\boldsymbol{u}}[\boldsymbol{b}]$ and $\mathbf{B} \models \bigwedge_{a \in A-B} \psi^{a,\boldsymbol{a}}_{x,\boldsymbol{u}}[\boldsymbol{b}]$,

if and only if $\mathbf{B} \models \varphi^{\boldsymbol{a}}_{\boldsymbol{u}}[\boldsymbol{b}]$. ⊠

Corollary. If P is an n-ary relation parametrically definable in $\mathbf{A}$, then $P \cap B^n$ is parametrically definable in $\mathbf{B}$.

Proof. Suppose $P(\boldsymbol{a})$ if and only if $\mathbf{A} \models \varphi[\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{a}]$, where $\boldsymbol{c}$ is a sequence of parameters in $A - B$ assigned to variables $\boldsymbol{u}$ in φ, and $\boldsymbol{d}$ a sequence of parameters in B. Then by the theorem, for any $\boldsymbol{b} \in B^n$, $P(\boldsymbol{b}) \Leftrightarrow \mathbf{A} \models \varphi[\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{b}] \Leftrightarrow \mathbf{B} \models \varphi^{\boldsymbol{c}}_{\boldsymbol{u}}[\boldsymbol{d}, \boldsymbol{b}]$. ⊠

Remark 1. Since B is a component, there are no relations between elements inside B and elements outside. We use this for the base of the induction. Nevertheless, we can do with a much weaker condition. All we need is the statement of the theorem for atomic formulas. That is, for every atomic formula $\alpha(u_1,\dots, u_k, v_1,\dots, v_l)$, there must be a finite choice of formulas $\psi(v_1,\dots, v_l)$ such that for every sequence $\boldsymbol{a} \in (A - B)^k$, there is some ψ satisfying for all $\boldsymbol{b} \in B^l$: $\mathbf{A} \models \alpha[\boldsymbol{a}, \boldsymbol{b}] \Leftrightarrow \mathbf{B} \models \psi[\boldsymbol{b}]$.

Remark 2. Equality may be viewed as a relation symbol, to be interpreted as the diagonal Δ of A; observe that $\Delta \subseteq B^2 \cup (A - B)^2$.

Remark 3. If there are constants (nullary operations), these must belong to B for the theorem to make sense. This rather compromises its applicability (see below).

Remark 4. The theorem continues to hold if $\mathcal{L}$ contains operation symbols of positive arity. Their interpretations (relations of a particular kind) must be contained in

$B^* \cup (A - B)^*$. To see that the proof goes through, assume operation symbols occur exclusively in atomic formulas of the form $x_0 = Qx_1 \ldots x_n$.

As stated, the theorem is trivial if there are operations of arity greater than 1, since there are no components other than A and $\emptyset$ in this case. It might still be of some use in the form suggested in the first remark.

Remark 5. The problems with operations stem from the requirement that they are everywhere defined.

2. Invariant Π^1_1-properties

Let $\mathbf{A}$ be a structure. A *decomposition* of $\mathbf{A}$ is a family $\langle \mathbf{B}_i \rangle_{i \in I}$ of components of $\mathbf{A}$ such that the system $\{B_i\}_{i \in I}$ is a partition of A. Such a decomposition is *definable* if for every index i there exist a formula $\beta_i(x, \boldsymbol{y}_i)$ and a sequence $\boldsymbol{a}_i$ of elements of $\mathbf{A}$ such that

$$B_i = \{b \in A \mid \mathbf{A} \models \beta_i[b, \boldsymbol{a}_i]\}.$$

A property $\mathcal{P}$ of structures in some class $\mathcal{K}$ is *invariant under decomposition* if for any structure $\mathbf{A} \in \mathcal{K}$, for every decomposition $\langle \mathbf{B}_i \rangle_{i \in I}$ of $\mathbf{A}$, $\mathbf{A}$ has $\mathcal{P}$ if and only if every $\mathbf{B}_i$ has $\mathcal{P}$. Analogously we have invariance under *definable* decomposition.

In his dissertation [D], Doets studied certain Π^1_1-properties of downwards linear orders that are invariant under decomposition. (An order is *downwards linear* if it satisfies $x \leq y \wedge z \leq y \rightarrow x \leq z \vee z \leq x$.) Examples of such properties are *completeness*, defined by

$$\forall X(\exists y \forall x(Xx \rightarrow y \leq x) \rightarrow \exists y \forall z(\forall x(Xx \rightarrow z \leq x) \leftrightarrow z \leq y)) \qquad (\mathbf{c})$$

and *well-foundedness*,

$$\forall X(\exists y\, Xy \rightarrow \exists y(Xy \wedge \forall z(Xz \wedge z \leq y \rightarrow y \leq z))) \qquad (\mathbf{wf})$$

If we want to catch a Π^1_1-property in first order axioms, a natural option is to turn the axiom defining it into a first order schema. A well-known example of this approach is the induction schema of first order Peano Arithmetic. Doets investigated whether, like Peano's induction axiom, (**wf**) is stronger than the corresponding first order schema (*definable well-foundedness*), in the sense of implying more first order sentences. Decompositions of orders come up repeatedly in the course of the investigation, and the question arises whether definable well-foundedness is invariant.

On a first order view, interpreting sentences such as (**c**) and (**wf**) involves a second universe, a universe of sets; an order $\mathbf{X}$ is well-founded in the standard sense if (**wf**) is satisfied in the structure $(\mathbf{X}, \mathcal{P}(X))$ that expands $\mathbf{X}$ with a second sort of individuals, the sets of individuals of the original universe X. (To be precise, there is also a relation of belonging involved, but we shall take that for granted.) In passing to definable well-foundedness, we replace the second sort by the collection Def($\mathbf{X}$) of parametrically definable subsets of X, i.e. the collection of all sets Y for which a formula φ exists and a sequence $\boldsymbol{x} \in X^*$ such that

$$Y = \{y \in X \mid \mathbf{X} \models \varphi[y, \boldsymbol{x}].$$

In general, we consider sorted structures ($\mathbf{A}$, $\mathcal{P}(A)$, $\mathcal{P}(A^2)$, $\mathcal{P}(A^3)$,…); and we let Def($\mathbf{A}$) denote the sequence of collections of definable n-ary relations, for $n = 1$, 2,… (These expansions with sorts look exactly like expansions with relations; what is meant, should always be apparent from the context.)

The reason why (**c**) and (**wf**) are invariant under decomposition is that their first order matrices are *local* in the following sense:

Definition. A first order matrix $\varphi(X_0,\ldots, X_{n-1})$ is *local* if for any suitable structure $\mathbf{A}$ and decomposition $\langle \mathbf{B}_i \rangle_{i \in I}$ of $\mathbf{A}$, for any relations $P_0, \ldots, P_{n-1}$ over A of the arities of $X_0,\ldots, X_{n-1}$ respectively,

$$(\mathbf{A}, P_0,\ldots, P_{n-1}) \models \varphi \Leftrightarrow \forall i \in I\, (\mathbf{B}_i, P_0 \cap B_i^*,\ldots, P_{n-1} \cap B_i^*) \models \varphi.$$

Theorem 2. Let φ be a Π^1_1 -sentence with local first order matrix. Then satisfaction of the first order schema corresponding with φ is invariant under definable decomposition.

Proof. Suppose $\varphi = \forall X_0\ldots X_{n-1}\psi(X_0,\ldots, X_{n-1})$, and $\langle \mathbf{B}_i \rangle_{i \in I}$ is a definable decomposition of $\mathbf{A}$.

Assume ($\mathbf{A}$, Def($\mathbf{A}$)) $\models \varphi$. Take any component $\mathbf{B} = \mathbf{B}_i$. Let $R_0,\ldots, R_{n-1}$ be definable relations over B, with the same arities as $X_0,\ldots, X_{n-1}$ respectively. Since B is definable, the R_j are definable in $\mathbf{A}$. Since ψ is local, $(\mathbf{B}, R_0,\ldots, R_{n-1}) \models \psi$. We may conclude that ($\mathbf{B}$, Def($\mathbf{B}$)) $\models \varphi$.

For the converse, assume ($\mathbf{B}_i$, Def($\mathbf{B}_i$)) $\models \varphi$ for each $i \in I$. Let $P_0, \ldots, P_{n-1}$ be suitable definable relations over A. Take any component $\mathbf{B} = \mathbf{B}_i$. By theorem 1, each $P_j \cap B^*$ is definable. So for every i, $(\mathbf{B}_i, P_0 \cap B_i^*,\ldots, P_{n-1} \cap B_i^*) \models \psi$. By locality, $(\mathbf{A}, P_0,\ldots, P_{n-1}) \models \psi$. We may conclude that ($\mathbf{A}$, Def($\mathbf{A}$)) $\models \varphi$. ⊠

Remark. For the proof of the theorem, *definable* locality, with the same definition as locality except that the range of the P_j is restricted to Def($\mathbf{A}$), is sufficient.

The application of the second theorem to definable completeness and well-foundedness in classes of orders is as follows. Assume that in a class $\mathcal{K}$ of orders there exists a bound N on the antichain complexity of components: if x and y belong to the same components, then there are $x_1, \ldots, x_N$ such that $x \leq x_1$, $x_1 \geq x_2, \ldots$, $x_{2k} \leq x_{2k+1}$, $x_{2k+1} \geq x_{2k+2}$, …, $x_N \leq y$. For example, for downwards linear orders $N = 2$. Then minimal components are parametrically definable. Since any component may be decomposed into minimal components, we get invariance, for the corresponding first order schemas, under arbitrary decompositions.

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Reference

[D] H.C. Doets: Completeness and definability. Dissertation, Amsterdam 1987.