Unary quantification redux

Bart Geurts & Jaap van der Does

Radboud University Nijmegen

It is well known that "most" is not first-order definable, and that the proof is in Barwise and Cooper's 1981 paper. Actually, Barwise and Cooper present two theorems that bear on the issue. Their theorem C12 says that, for any pair of one-place predicates A and B, there is no sentence of classical predicate logic that is true iff "Most A are B" is. (Let's assume that "Most A are B" means that more than half of the A's are B, though the only thing that matters is that "most" is proportional.) Barwise and Cooper's C13 states that the foregoing result remains valid when classical logic is enriched with a unary quantifier Q so defined that Qx(Ax) is true iff more than half of the entities in the domain of quantification are A's.

There is a popular legend about the import of these theorems. It originated in Barwise and Cooper's own discussion (especially section 1.2, pp. 160-161), soon became lore, and has meanwhile made it into the textbooks. Here is how your favourite introduction to logic and semantics puts it:

> Quantifiers like *many* and *most* are essentially two-place. This can be proved rigorously, but we will not do so here. Such expressions must, then, be interpreted as expressions of type $\langle\langle e,t\rangle,\langle\langle e,t\rangle,t\rangle\rangle$, that is, as two-place second-order relations. (Gamut 1991: 116)

In the same vein, an eminent philosopher of language teaches his students that:

> There are perfectly precise binary quantifiers that cannot be defined in terms of unary quantifiers. A paradigm example is *most*, interpreted as meaning *more than half*. (MacFarlane 2011: 3)

According to the legend, Barwise and Cooper proved that "most" is not a unary quantifier. It is plain, however, that this doesn't follow from Barwise and Cooper's results. What Barwise and Cooper showed is that classical predicate logic lacks the expressive power for capturing the meaning of "most", even if we extend that logic with a quantifier that allows us to make statements about most individuals. The claim that there is no way of representing "most" as a unary quantifier goes beyond that. To obtain the non-definability results it must be assumed there is no fixed finite number of things, or else a brute force disjunctive enumeration of the relevant cases will do. Only if there is no upper bound on the finite number of things, such definitions are no longer available; then any purported first-order definition of "most" fails.

Despite the ingenuity of the proofs, one wonders what bearing their idealizations might have on our understanding natural language quantification. In what way is the (in)ability of a logic to define a quantifier relevant? As Barwise and Cooper themselves remark, it shows first-order logic is not rich enough to capture all aspects of natural language semantics. From this it just follows we should consider other logics. But then the discrepancy between the legend and Barwise and Cooper's theorems may well disappear. In this note we consider the logic in Belnap (1970), who proposes a system for studying conditional assertion and suggests that it can be used to restrict unary quantifiers in a uniform way.

Belnap aims to capture the insight that the affirmation of a form "If p then q" is not so much an affirmation of a conditional as a conditional affirmation of the consequent. When affirming "If p then q", we only commit ourselves to q on the condition that p is true. Belnap writes "p/q" for the conditional assertion, and observes that it can be used to restrict first-order quantifiers in a uniform way. Belnap's system for conditional assertion combines assertability conditions and truth conditions as follows:

> The sentence p/q is assertable iff p is true. If p/q is assertable, p/q is true iff q is. (Belnap 1970: 3)

To get a feel for the distinctive features of Belnap's system, we present a simplified version. For our current purposes it suffices to combine assertability and truth conditions (rather than intensionality). Also, rather than using substitution, we obtain propositions as special cases of formulas and think of asserting an open formula as asserting the proposition with the free variables x referring to the values $g(\mathrm{x})$ assigned to it by g. (As usual, $g[\mathrm{x}/a]$ is that variable assignment which assigns x to a and is otherwise identical to g.)

1. If φ is atomic, then $\|\varphi\|_g$ is assertable, and $\|\varphi\|_g = 1$ iff the usual conditions apply.
2. $\|\neg\varphi\|_g$ is assertable iff $\|\varphi\|_g$ is; if assertable, $\|\neg\varphi\|_g = 1$ iff $\|\varphi\|_g = 0$.
3. $\|\varphi \wedge \psi\|_g$ is assertable iff $\|\varphi\|_g$ or $\|\psi\|_g$ is; if assertable, $\|\varphi \wedge \psi\|_g = 1$ iff $\|\varphi\|_g = 1$ and $\|\psi\|_g = 1$.
4. $\|\varphi \vee \psi\|_g$ is assertable iff $\|\varphi\|_g$ or $\|\psi\|_g$ is; if assertable, $\|\varphi \vee \psi\|_g = 1$ iff $\|\varphi\|_g = 1$ or $\|\psi\|_g = 1$.
5. $\|\varphi/\psi\|_g$ is assertable iff $\|\varphi\|_g = 1$; if assertable, $\|\varphi/\psi\|_g = \|\psi\|_g$.

6. $\|\forall x\varphi\|_g$ is assertable iff $\|\varphi\|_{g[x/a]}$ is assertable for some a; if assertable, $\|\forall x\varphi\|_g = 1$ iff $\|\varphi\|_{g[x/a]} = 1$ for all a for which $\|\varphi\|_{g[x/a]}$ is assertable.
7. $\|\exists x\varphi\|_g$ is assertable iff $\|\varphi\|_{g[x/a]}$ is assertable for some a; if assertable, $\|\exists x\varphi\|_g = 1$ iff $\|\varphi\|_{g[x/a]} = 1$ for at least one a for which $\|\varphi\|_{g[x/a]}$ is assertable.

In a way, it is a bit awkward to speak of assertability in the context of an extensional semantics, and Belnap too felt that a purely semantic approach leaves out crucial pragmatic aspects of assertability. Our extensional simplification is even more radical in this regard, and barely justifies speaking of "assertability". Perhaps, then, "definedness" would be a better term. However, we will stick to Belnap's terminology, if only to stay connected with his work.

Note that assertability conditions may be weaker than expected. In particular, a conjunctive complex with "$\wedge$" or "$\forall$" is assertable iff at least one of its conjuncts is, and if assertable its assertable parts alone determine whether or not the complex is true. Also, the distinction between assertability and truth-conditions only plays a rôle in sentences containing a conditional assertion; otherwise any sentence is assertable and has its usual truth-conditions.

Having presented his system, Belnap observes that inserting φ/ψ in the immediate scope of a quantifier has the effect of restricting the domain of quantification to individuals of which φ holds. Thus, if "All crows are black" is symbolised as $\forall$x(Cx/Bx), then stating this sentence has the same effect as stating $\forall$x(Bx) with the domain of quantification restricted to crows. And, much to Belnap's own surprise and delight, the same holds for the existential quantifier:

> Having unexpectedly obtained restricted universal quantification by artless combination, one wonders about restricted existential quantification. [...] After puzzling for a while, one eventually remembers how hard it is to teach freshmen not to render "Some crows are black" by "$\exists$x(if Cx then Bx)". Then, thinking of these foolish freshmen, feeling foolish oneself, and hoping no one is looking, one finally writes down $\exists$x(Cx/Bx) as a possible reading of "Some crows are black." And Aristotle be Russell if it doesn't turn out that this amounts exactly to that reading of the Aristotelian I-form which makes "Some crows are black" amount to "Consider the crows: some of them are black." (Belnap 1970: 7-8)

This elegant uniformity makes one wonder whether Belnap's idea of domain-restriction would work as well for non-classical quantifiers. We believe it does. Somewhat informally, "most" could be represented as follows:

8. $\|Mx\varphi\|_g$ is assertable iff $\|\varphi\|_{g[x/a]} = 1$ is for some a; if assertable, $\|Mx\varphi\|_g =$

1 iff $\|\varphi\|_{g[\mathrm{x}/a]} = 1$ for more than half of the individuals a for which $\|\varphi\|_{g[\mathrm{x}/a]}$ is assertable.

Thus, "Most swans are white" is rendered as "Mx(Sx/Wx)", which is true iff more than half of the swans are white. Observe that in this setting "most" is construed as a unary quantifier, which refutes the popular opinion about the binary essence of "most" and kindred expressions.

Before we continue to give the observation in its most general form and demonstrate its correctness, let us first check some examples to see how Belnap's analysis effects domain restriction:

9. *All crows are black:*
   Assume there are crows. Then "∀x(Cx/Bx)" is assertable, and it is true iff for all a for which $\|\mathrm{Cx/Bx}\|_{g[\mathrm{x}/a]}$ is assertable, $\|\mathrm{Cx/Bx}\|_{g[\mathrm{x}/a]} = 1$; iff for all a for which $\|\mathrm{Cx}\|_{g[\mathrm{x}/a]} = 1$, $\|\mathrm{Bx}\|_{g[\mathrm{x}/a]} =1$; iff $\|\forall\mathrm{x(Cx,Bx)}\|_g = 1$.
10. *Some crows are black:*
    Assume there are crows. Then "∃x(Cx/Bx)" is assertable, and it is true iff for some a with $\|\mathrm{Cx/Bx}\|_{g[\mathrm{x}/a]}$ assertable, $\|\mathrm{Cx/Bx}\|_{g[\mathrm{x}/a]} = 1$; iff for some a for which $\|\mathrm{Cx}\|_{g[\mathrm{x}/a]} =1$, $\|\mathrm{Bx}\|_{g[\mathrm{x}/a]} = 1$; iff $\|\exists\mathrm{x(Cx,Bx)}\|_g = 1$.
11. *Most crows are black:*
    Assume there are crows. Then, "Mx(Cx/Bx)" is assertable, and it is true iff the a for which $\|\mathrm{Cx/Bx}\|_{g[\mathrm{x}/a]}$ is assertable and $\|\mathrm{Cx/Bx}\|_{g[\mathrm{x}/a]} = 1$ outnumber the a' for which $\|\mathrm{Cx/Bx}\|_{g[\mathrm{x}/a']}$ is assertable and $\|\mathrm{Cx/Bx}\|_{g[\mathrm{x}/a']} = 0$; iff there are more a for which $\|\mathrm{Cx} \wedge \mathrm{Bx}\|_{g[\mathrm{x}/a]} = 1$ than a' for which $\|\mathrm{Cx} \wedge \neg\mathrm{Bx}\|_{g[\mathrm{x}/a']}$ $= 1$; iff $\|\mathrm{Mx(Cx,Bx)}\|_g = 1$.

To generalise Belnap's observation, we refer to Mostowski (1957), who defines a quantifier as a functor that assigns to each domain E a logical $\mathrm{Q_E} \subseteq \mathcal{P}(\mathrm{E})$. (A quantifier is logical iff, for all X and Y: if X ∈ Q and |X| = |Y|, then Y ∈ Q.) To begin with, we need the appropriate notion of a definable set, which acts as domain and argument of the quantifier. Sets (properties) are not assertable, rather they are assertable parts:

12. $\|\lambda\mathrm{x}.\varphi\|_g^{\mathrm{E}}$ is an assertable part iff $\|\varphi\|_{g[\mathrm{x}/a]} = 1$ for some $a \in \mathrm{E}$; if an assertable part, $\|\lambda\mathrm{x}.\varphi\|_g^{\mathrm{E}} = \{a \in \mathrm{E} \mid \|\varphi\|^{\mathrm{E}}_{g[\mathrm{x}/a]}$ is assertable and $\|\varphi\|^{\mathrm{E}}_{g[\mathrm{x}/a]} = 1\}$.

Note the domain E is made explicit. A first attempt at defining the quantifier case is:

13. $\|\mathrm{Qx}\varphi\|_g^{\mathrm{E}}$ is assertable iff $\|\varphi\|^{\mathrm{E}}_{g[\mathrm{x}/a]} = 1$ is for some $a \in \mathrm{E}$; if assertable, $\|\mathrm{Qx}\varphi\|_g^{\mathrm{E}}$

$= 1$ iff $\|\lambda x.\varphi\|^E_g \in Q_E$.

However, this definition does not yet yield the truth-conditions we are after. To see this, consider $\|\forall x(Cx/Bx)\|^E_g$, and assume there are crows. Then, $\|\forall x(Cx/Bx)\|^E_g = 1$ iff $\|\lambda x.Cx/Bx\|^E_g \in Q_E$. Writing C for $\|\lambda x.Cx\|^E_g$ and B for $\|\lambda x.Bx\|^E_g$, we have: $\|\forall x(Cx/Bx)\|^E_g = 1$ iff $C \cap B = E$. That is, everything is a black crow. But of course, "All crows are black" should allow for non-black non-crows. The point is: in finding the most general formulation the appropriate domain restriction has not yet been included. According to Belnap, we are after a formalization of: 'Consider the crows, they are all black'. To this end, we suggest treating $\|Qx(\varphi/\psi)\|^E_g$ as a special case in which conditional assertion induces a kind of "domain dynamics":

14. $\|Qx(\varphi/\psi)\|^E_g$ is assertable iff $\|\varphi\|^E_{g[x/a]} = 1$ is for some $a \in E$; if assertable, $\|Qx(\varphi/\psi)\|^E_g = 1$ iff $\|\lambda x.\varphi/\psi\|^E_g \in Q_{\|\lambda x.\varphi\|^E_g}$.

Assuming there are crows, we now have (in shorthand):

15. *All crows are black:*
$\forall_E(C/B)$ iff $\forall_C(C \cap B)$ iff $C \cap B = C$ iff $C \subseteq B$

16. *Some crows are black:*
$\exists_E(C/B)$ iff $\exists_C(C \cap B)$ iff $C \cap B \neq 0$

17. *Most crows are black:*
$M_E(C/B)$ iff $M_C(C \cap B)$ iff $|C \cap B| > |C - (C \cap B)|$ iff $|C \cap B| > |C - B|$

This is all as it should be. But it does seem that in its most general form Belnap's observation comes at a prize: The side-effects of conditional assertion depend on the context in which it occurs, and so is not compositional. In other words, if compositionality is required, conditional assertion does not yield the intended result; domain restriction has to be enforced separately. Still, it remains true that to specify the truth-conditions of natural language quantification, domain restricted unary quantification combined with conditional assertion suffices. There is nothing "essentially two-place" about "most".

The modified Belnap observation dates back to Westerståhl (1985), who analysed domain restriction in terms of two well known quantifier universals (see also: Westerståhl 1989, theorem 3.2.3, or Peters and Westerståhl 2006, §4.5). In natural language two-place monadic quantification is a very common form, and it has been argued to satisfy both EXT and CONS:

EXT: $Q_E(A, B)$ iff $Q'_E(A, B)$, for all $E, E' \supseteq A, B$.

CONS: $Q_E(A, B)$ iff $Q_{E'}(A, A \cap B)$, for all $E \supseteq A, B$.

Westerståhl observes that every unary quantifier Q can be relativised to a predicate A thus:

$$Q^r_E(A, B) \text{ iff } Q_A(A \cap B)$$

Clearly, Q^r satisfies EXT and CONS. But also conversely, a two-place quantifier Q' gives rise to a unary quantifier:

$$(Q')^u_E(A) \quad \text{iff} \quad Q'_E(E, B)$$

It is easy to see that if Q' satisfies EXT and CONS: $((Q')^u)^r_E = Q'_E$. In this sense, natural language two-place monadic quantifiers can always be reduced to monadic quantification; i.e. Belnap's suggestion. From a logical point of view Westerståhl's observation may look shallow compared to Barwise and Cooper's expressibility results. But as far as natural language is concerned, it is more to the point and deserves a more prominent place in the textbooks.

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