QUANTUM LOGIC, QUANTUM HISTORIES AND QUANTUM TURING MACHINES

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1 Introduction

In his rich research Johan van Benthem has given some important contributions to temporal and dynamic logics. In this paper, we will present the main ideas of a semantic framework, where quantum logic permits us to model some typical temporal and dynamic situations. One is dealing with new quantum logical structures that have been suggested by the consistent histories approach to quantum theory (investigated by Gell-Mann, Hartle ([4]), Isham, Linden ([7], [8]) and many others).

Roughly, the basic idea of this approach is that the fundamental objects of quantum theory are represented by consistent sets of histories, where each history corresponds to a possible temporal evolution of a micro-physical system. Interestingly enough, one can find deep connections between quantum histories and quantum computations.

We propose a general semantics that, in principle, admits a number of possible applications. The basic concept in this semantics is the notion of historical structure.

2 Historical structures

Definition 2.1 A historical structure is structure
\[ \mathcal{M} = \langle T, S, Ev, Histev, Op, D \rangle, \]
where

1) \( T \) is a linearly ordered set of times. In the following a time-sequence \( \langle t_1, \ldots, t_n \rangle \) will be always written according to the order \( \leq \).

2) \( S \) is a function that assigns to each \( t \in T \) a set \( S_t \) of possible states at time \( t \). For simplicity all \( S_t \) are supposed to be (ontologically distinct) copies of a fixed (timeless) \( S^* \).

\[ \text{From an intuitive point of view, states can be regarded as possible worlds of a Kripke-style semantics: pieces of information about possible states of affairs (or fragments of reality).} \]

In the physical applications: \( S^* \) will (naturally) contain the possible states of the physical system under investigation. A state is called pure when it represents a maximal information that is consistent with respect to the theory. Non maximal pieces of information correspond to mixed states. In classical mechanics pure states are always logically complete. In other words any pure state can semantically decide any relevant physical property of the system (localization, velocity, energy, ...). On the contrary, a characteristic aspect of quantum theory is a divergence between maximality and logical completeness. Owing to the uncertainty relations, any information about a quantum system is necessarily logically incomplete, in the sense that it cannot semantically decide all the physical properties concerning our system. In classical mechanics (CM), pure states are
mathematically represented by points of an appropriate phase space $\Omega$. In quantum theory (QT), instead, the pure states of a quantum object are represented by unitary vectors in an appropriate Hilbert space $\mathcal{H}$. 

3) $Ev$ is a function that assigns to each $t \in T$ a set $\mathcal{E}_v_t$ of events at time $t$. For simplicity, all $\mathcal{E}_v_t$ are supposed to be (ontologically distinct) copies of a fixed (timeless) $\mathcal{E}_v^*$. 

In the case of our physical applications: $\mathcal{E}_v^*$ will contain the possible physical properties of the physical system. In CM: measurable subsets of the phase space $\Omega$. In QT: closed subspaces (or equivalently, projection operators) in the appropriate Hilbert space $\mathcal{H}$. 

Any timeless state $s$ will assign to any timeless event $\alpha$ a value in the interval $[0,1]$: 

$$s(\alpha) \in [0,1].$$

A state $s$ is said to verify an event $\alpha$ ($s \models \alpha$) iff $s(\alpha) = 1$. Similarly for all states in $S_t$ and all events in $\mathcal{E}_v_t$. 

The set $\mathcal{E}_v^*$ of the timeless events has a structure (for simplicity all $\mathcal{E}_v_t$ are assumed to be isomorphic to $\mathcal{E}_v^*$). We will consider only examples of structures that are partially ordered by a relation $\sqsubseteq$. 

In the case of CM, $\mathcal{E}_v^*$ gives rise to a Boolean algebra (a convenient subalgebra of the power-set of $\Omega$). Similarly, in QT, the structure of $\mathcal{E}_v^*$ can be identified with the algebra of all closed subspaces of $\mathcal{H}$ (a particular example of a non distributive orthomodular lattice). As a consequence one obtains that, differently from the classical mechanical case, quantum events do not have a classical logical behaviour. 

We will call historical sequence a sequence of events $\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle$, where each $\alpha_{t_i}$ is in $\mathcal{E}_v_{t_i}$. Of course, the algebraic structure of each $\mathcal{E}_v_{t_k}$ can be naturally transferred to the set of all historical sequences ($\prod_{i=1}^{n} \mathcal{E}_v_{t_i}$). We will call temporal support ([7]) of the historical sequence $\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle$, the time-sequence $\langle t_1, \ldots, t_n \rangle$. Composition between temporal supports and historical sequences with disjoint temporal supports is defined in the expected way. 

4) $Histev$ is a function that assigns to each time-sequence $\langle t_1, \ldots, t_n \rangle$ the set of the historical events $Histev^{(t_1, \ldots, t_n)}$ at time $\langle t_1, \ldots, t_n \rangle$. This set is equipped with a structure. 

Physical examples: 

In CM: $Histev^{(t_1, \ldots, t_n)}$ is a $\sigma$-field of subsets of the cartesian product of $S_{t_1} \times \ldots \times S_{t_n}$. 

In other words, a historical event (at time $\langle t_1, \ldots, t_n \rangle$) is a set of sequences $\{s_{t_1}, \ldots, s_{t_n}\}$, where any $s_{t_i}$ is a pure state of the system at time $t_i$. 

3
In QT: \( \text{Histev}^{(t_1, \ldots, t_n)} \) can be identified with the set of the closed subspaces of the tensor product

\[
\mathcal{H}_{t_1} \otimes \ldots \otimes \mathcal{H}_{t_n},
\]

where each \( \mathcal{H}_{t_i} \) represents the Hilbert space of the system at time \( t_i \).

Any historical sequence of events must be represented by a historical event. However, not all historical events will represent historical sequences. For instance, in QT the sequence \( \langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle \) (where each \( \alpha_{t_i} \) is a closed subspace) will be represented by the tensor product \( \alpha_{t_1} \otimes \ldots \otimes \alpha_{t_n} \). Such a product is a closed subspace in the tensor-product space \( \mathcal{H}_{t_1} \otimes \ldots \otimes \mathcal{H}_{t_n} \). Of course, not all the closed subspaces in the tensor-product space will have this factorized form. For instance, the orthocomplement of a factorized historical event \( \alpha_{t_1} \otimes \ldots \otimes \alpha_{t_n} \) (which represents the logical negation of the original event) will not generally correspond to any historical sequence of events. Our physical examples naturally suggest to require the following general conditions:

(4.1) For any time \( t_k \) and any time-sequence \( \langle t_1, \ldots, t_k, \ldots, t_n \rangle \), there exists a function \( f \) that maps \( \mathcal{E}v_{t_k} \) into \( \text{Histev}^{(t_1, \ldots, t_n)} \):

\[
f : \mathcal{E}v_{t_k} \to \text{Histev}^{(t_1, \ldots, t_n)}.
\]

Further, such \( f \) is an embedding that preserves the algebraic structure of \( \mathcal{E}v_{t_k} \).

This guarantees that any event at time \( t_k \) is represented by a historical event in any longer time-interval. For instance, in the case of closed subspaces, \( f(\alpha_{t_k}) \) will be \( 1_{t_1} \otimes \ldots \otimes \alpha_{t_k} \otimes \ldots \otimes 1_{t_n} \), where \( 1_{t_i} \) represents the certain event at time \( t_i \).

(4.2) For any time sequence \( \langle t_1, \ldots, t_n \rangle \), there is a function \( g \) that transforms any historical sequence \( \langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle \) of events into a historical event \( \eta^{(t_1, \ldots, t_n)} \) of \( \text{Histev}^{(t_1, \ldots, t_n)} \):

\[
g : \langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle \mapsto \eta^{(t_1, \ldots, t_n)}.
\]

Further, such \( g \) is a homomorphism of \( \prod_{i=1}^n \mathcal{E}v_{t_i} \) into \( \text{Histev}^{(t_1, \ldots, t_n)} \).

**Definition 2.2** Two historical sequences \( \langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle \) and \( \langle \beta_{t_1}, \ldots, \beta_{t_n} \rangle \) are called equivalent \( \langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle \approx \langle \beta_{t_1}, \ldots, \beta_{t_n} \rangle \) iff \( g(\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle) = g(\langle \beta_{t_1}, \ldots, \beta_{t_n} \rangle) \).

Since \( g \) is a homomorphism, the equivalence relation \( \approx \) turns out to be a congruence on the algebraic structure induced on \( \prod_{i=1}^n \mathcal{E}v_{t_i} \). Further, the relation \( \approx \) is required to satisfy the following condition:
let \(\langle t_m, \ldots, t_n \rangle\) and \(\langle t_i, \ldots, t_j \rangle\) be two disjoint temporal supports: if 
\(\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle \approx \langle \alpha'_{t_m}, \ldots, \alpha'_{t_n} \rangle\) and \(\langle \beta_{t_i}, \ldots, \beta_{t_j} \rangle \approx \langle \beta'_{t_i}, \ldots, \beta'_{t_j} \rangle\), then 
\(\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle \circ \langle \beta_{t_i}, \ldots, \beta_{t_j} \rangle \approx \langle \alpha'_{t_m}, \ldots, \alpha'_{t_n} \rangle \circ \langle \beta'_{t_i}, \ldots, \beta'_{t_j} \rangle\).

In other words, the relation \(\approx\) is preserved under composition \((\circ)\) of historical sequences.

Any historical event \(\eta^{(t_1, \ldots, t_n)}\) that represents (via \(g\)) a historical sequence will be called a history \(^1\). We will indicate by \(Hist^{(t_1, \ldots, t_n)}\) the subset of \(Histev^{(t_1, \ldots, t_n)}\) that contains all the histories. Instead of \(g(\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle)\) we will simply write: \(\eta^{\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle}\).

On this basis one can naturally define a notion of historical truth. This is a semantic notion that may hold between a sequence of states and a historical event. Let us first consider the case where both the state-sequence and the historical event refer to the same temporal support \(\langle t_1, \ldots, t_n \rangle\).

**Definition 2.3 Restricted definition of historical truth**

\((\langle s_{t_1}, \ldots, s_{t_n} \rangle\) verifies \(\eta: \langle s_{t_1}, \ldots, s_{t_n} \rangle \models \eta)\)

We will distinguish the case of histories from that of historical events that do not represent histories.

a) Let \(\eta\) be the history \(\eta^{\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle}\). Then:
\(\langle s_{t_1}, \ldots, s_{t_n} \rangle \models \eta\) if for any sequence \(\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle\) s.t.
\(g(\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle) = \eta\): \(s_{t_i} \models \alpha_{t_i}\), for any \(s_{t_i}\) and \(\alpha_{t_i}\) \((1 \leq i \leq n)\).

b) Let \(\eta\) be a historical event (belonging to \(Histev^{(t_1, \ldots, t_n)}\)) that is not a history. Then:
\(\langle s_{t_1}, \ldots, s_{t_n} \rangle \models \eta\) if for at least one history \(\delta^{\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle}\):

i) \(\delta^{\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle} \sqsubseteq \eta\), where \(\sqsubseteq\) is the partial order of the event structure;

ii) \(\langle s_{t_1}, \ldots, s_{t_n} \rangle \models \delta^{\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle}\).

Our truth definition can be naturally extended also to the case where a state-sequence and a historical event refer to different time-sequences. Let us first introduce a procedure that permits us to normalize any history to a given time-sequence.

**Definition 2.4 Normalization of a historical-sequence to a given time sequence**

Let \(\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle\) be a historical sequence and let \(\langle t_i, \ldots, t_j \rangle\) be any time-sequence. The normalization of \(\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle\) to \(\langle t_i, \ldots, t_j \rangle\) is the following historical sequence:

\([\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle]_{\langle t_i, \ldots, t_j \rangle} := \langle \beta_{t_i}, \ldots, \beta_{t_j} \rangle\),

\(^1\)In \([7]\), historical sequences are called *history-filters*, while historical events are called *history-propositions*.
where
\[ \beta_{t_k} = \begin{cases} \alpha_{t_k}, & \text{if } t_k \text{ is in } \langle t_m, \ldots, t_n \rangle; \\ 1_{t_k} & \text{otherwise.} \end{cases} \]

(1_{t_k} is the certain event at time t_k).

Accordingly, the normalization of a history will be automatically determined (via g):

**Definition 2.5** Normalization of a history to a given time-sequence
Let \( \eta^{(\alpha_{t_m}, \ldots, \alpha_{t_n})} \) be a history and \( \langle t_i, \ldots, t_j \rangle \) be a time-sequence. The normalization of \( \eta^{(\alpha_{t_m}, \ldots, \alpha_{t_n})} \) to \( \langle t_i, \ldots, t_j \rangle \) is the history
\[ \lceil \eta^{(\alpha_{t_m}, \ldots, \alpha_{t_n})} \rceil \langle t_i, \ldots, t_j \rangle \]
that is univocally determined by the historical sequence
\[ \lceil (\alpha_{t_m}, \ldots, \alpha_{t_n}) \rceil \langle t_i, \ldots, t_j \rangle. \]

One can easily prove that Definition 2.5 is a good definition since it is independent of the choice of the representative.

On this basis we can define a general notion of historical truth:

**Definition 2.6** General definition of historical truth
(\( \langle s_{t_1}, \ldots, s_{t_n} \rangle \) verifies \( \eta^{(t_i, \ldots, t_j)} \): \( \langle s_{t_1}, \ldots, s_{t_n} \rangle \models \eta^{(t_i, \ldots, t_j)} \))
Let \( \eta^{(t_i, \ldots, t_j)} \) be a historical event. \( \langle s_{t_1}, \ldots, s_{t_n} \rangle \models \eta^{(t_i, \ldots, t_j)} \) iff there is a history \( \delta^{(t_i, \ldots, t_j)} \) s.t.:
1) \( \delta^{(t_i, \ldots, t_j)} \subseteq \eta^{(t_i, \ldots, t_j)} \);
2) \( \langle s_{t_1}, \ldots, s_{t_n} \rangle \models \lceil \delta^{(t_i, \ldots, t_j)} \rceil \langle t_1, \ldots, t_n \rangle. \)

A historical sequence will be called normal when all historical events can be normalized to any time sequence.

It turns out that any historical structure where all \( Hist_{E}^{(t_1, \ldots, t_n)} \) are structured as complete lattices is trivially normal.

5) \( Op \), the operation function, is a function that associates to any pair of time \( t_i, t_j \) a set of operations \( O_{t_i}^{t_j} \)
\[ Op : \langle t_i, t_j \rangle \mapsto O_{t_i}^{t_j}, \]
where \( O_{t_i}^{t_j} \) is a subset of a set of admissible operations \( O \subseteq \{ f \mid S^* \to S^* \} \) \( (S^* \subseteq S^*) \). In other words, admissible operations transform states into states.
Some operations in $O_{t_i}^j$ may represent spontaneous evolutions whereas other operations may represent state transformations induced by an action. A typical action is a test performed in order to check whether a certain property $\alpha$ holds\(^2\). Of course, physical measurements are paradigmatic examples of action of this kind. We will indicate by $f^\alpha(s)$ the transformation of a state $s$ induced by an action that has been performed in order to check whether the object in state $s$ satisfies the property $\alpha$.

On this basis a class of accessibility relations can be naturally defined in terms of our operations (differently from Kripke semantics, where accessibility relations are usually dealt with as primitive). Let $s,u$ represent timeless states.

States $s$ and $u$ are called accessible in the time interval $[t_i,t_j]$ (we will write $Acc_{t_i}^{t_j}(s,u)$) iff there exists an operation $f \in O_{t_i}^j$ s.t.

$$u = f(s).$$

States $s$ and $u$ are absolutely accessible ($Acc(s,u)$) iff for at least two times $t_i,t_j$:

$$Acc_{t_i}^{t_j}(s,u).$$

Let $\alpha$ be an event. The states $s,u$ are $\alpha$-accessible ($Acc^\alpha(s,u)$) iff there exists an operation $f^\alpha$ s.t.

$$u = f^\alpha(s).$$

Since all $S_t$ are copies of $S^*$ our accessibility relations are automatically transferred to pairs of states that may belong to different $S_t$.

6) $D$ is a (possibly empty) set of decoherence functionals $d$ ([7]). From the intuitive point of view, $d(\eta,\delta)$ measures the degree of interference between the historical events $\eta$ and $\delta$.

### 3 Dynamic and temporal operators

Different logical operators that have a temporal or a dynamic meaning can be naturally defined in our semantics.

**Definition 3.1** The temporal conjunction and then $\sqcap \leadsto$

Let $\eta_{\langle t_m,\ldots,t_n \rangle},\delta_{\langle t_i,\ldots,t_j \rangle}$ be two historical events. Generally $\sqcap \leadsto$ is a partial operation that turns out to be always defined in the particular case of a normal historical structure where any $Histev_{\langle t_1,\ldots,t_n \rangle}$ gives rise to a lattice (where $\sqcap$ and $\sqcup$ represent the infimum and the supremum, respectively).

$$\eta_{\langle t_m,\ldots,t_n \rangle} \sqcap \leadsto \delta_{\langle t_i,\ldots,t_j \rangle} := \begin{cases} 0_{\langle t_m,\ldots,t_n \rangle \circ \langle t_i,\ldots,t_j \rangle}, & \text{if } \langle t_m,\ldots,t_n \rangle \text{ does not precede } \langle t_i,\ldots,t_j \rangle \text{ w.r.t the order } \leq; \\ [\eta]_{\langle t_m,\ldots,t_n \rangle \circ \langle t_i,\ldots,t_j \rangle} \cap [\delta]_{\langle t_m,\ldots,t_n \rangle \circ \langle t_i,\ldots,t_j \rangle}, & \text{if this inf exists in } Histev_{\langle t_m,\ldots,t_n \rangle \circ \langle t_i,\ldots,t_j \rangle}; \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

\(^2\)See von Wright [14].
As an example, suppose two histories \( \eta^{(\alpha_{t_1},\alpha_{t_2})} \), \( \delta^{(\beta_{t_3},\beta_{t_4})} \) \((t_1 < t_2 < t_3 < t_4)\).

According to our definition we will have:

\[
\eta^{(\alpha_{t_1},\alpha_{t_2})} \sqcap \neg \delta^{(\beta_{t_3},\beta_{t_4})} = \eta^{(\alpha_{t_1},\alpha_{t_2},1_{t_3},1_{t_4})} \sqcap \delta^{(1_{t_1},1_{t_2},\beta_{t_3},\beta_{t_4})} = \gamma^{(\alpha_{t_1},\alpha_{t_2},\beta_{t_3},\beta_{t_4})}.
\]

We obtain in this way the expected meaning of a temporal conjunction: \( \eta^{(\alpha_{t_1},\alpha_{t_2})} \)
and then \( \delta^{(\beta_{t_3},\beta_{t_4})} \) is the history determined by the historical sequence
\( (\alpha_{t_1},\alpha_{t_2},\beta_{t_3},\beta_{t_4}) \). 

A dynamic implication

Let us refer to set \( E^v \) of all timeless events and to the set \( S^s \) of all timeless states. Let \( \alpha \in E^v \). We define first a function \( \alpha \) that assign to each \( \beta \in E^v \) a set of timeless states:

\[
\alpha : E^v \rightarrow P(S^s) \quad (\text{where } P(S^s) \text{ is the power-set of } S^s)
\]

satisfying the condition:

\[
s \in \alpha \beta \iff \forall u [\text{Acc}\alpha(s,u) \Rightarrow u \models \beta].
\]

From the intuitive point of view, \( s \in \alpha \beta \) has the following meaning: suppose we test \( \alpha \) on a state \( s \) and we obtain a positive result; then, any state into which \( s \) is transformed after such a test verifies \( \beta \). Of course, \( \alpha \beta \) does not necessarily determine an event. In the case where \( E^v \) has the structure of a complete lattice, one can easily define the following total operation:

\[
\alpha : E^v \rightarrow E^v,
\]

where

\[
\alpha \beta := \bigcap \{ \delta \mid \forall u \in \alpha \beta : u \models \delta \}
\]

Otherwise, \( \alpha \beta \) will represent a partial operation.

In the particular case of the lattice of all closed subspaces in a Hilbert space \( \mathcal{H} \) the dynamic implication is always defined and corresponds to the usual quantum logical (material) implication (which is also called Sasaki implication).

As is well known, differently from classical logic, \( \alpha' \sqcup \beta \) (where \( \sqcup \) is the orthocomplement) does not represent a good conditional operation in quantum logic. Actually, it may happen that in a given orthomodular lattice, \( \alpha' \sqcup \beta \) is equal to \( 1 \), even if \( \alpha \) does not precede \( \beta \) according to the lattice-order. From the logical point of view, this means that a sentence like “not A or B” might be true in a given (algebraic) model, even if \( A \) does not imply \( B \) in the same model.

However, a particular variant of \( \alpha' \sqcup \beta \) (which is equivalent to \( \alpha' \sqcup \beta \) in all Boolean lattices) permits us to define a good quantum logical conditional. It is sufficient to put:

\[
\alpha \rightarrow \beta := \alpha' \sqcup (\alpha \sqcap \beta).
\]
One can easily show that for closed subspaces $\alpha, \beta$:

\[ \text{Acc}^\beta(\alpha) = \alpha \rightarrow \beta. \]

Of course, such a relation holds, only if we interpret the accessibility relation $\text{Acc}^\alpha$ in the “natural” quantum theoretical way. In other words:

$\text{Acc}^\alpha(s, u)$ iff $u$ is the result of the projection of $s$ over the closed subspace $\alpha$.

This shows that the standard logical implication admits a natural dynamic interpretation. In other words, even the orthodox version of quantum logic (which is usually described as a “static logic”) seems to have some "hidden" dynamic features.

4 Quantum histories and quantum Turing machines

Let us first introduce a definition of quantum Turing machine, following a quite elegant abstract approach, that has been recently proposed by Gudder ([5]).

**Definition 4.1** A quantum Turing machine, whose tape is identified with the set of the integers $\mathbb{Z}$, is a structure

\[ M = \langle I, S, \delta, \mathcal{H} \rangle \]

where:

1) $I$ is a finite alphabet, containing a blank symbol $\#$.

2) $S$ is a set of memory states, containing an initial state $s_0$ and a final state $s_f$. Similarly to classical Turing machines, consider the cartesian product $S \times I \times I \times \{L, R\} \times S$.

The interpretation of any element $\langle s, x, y, d, r \rangle$ (where $d \in \{R, L\}$) of our product will be the following: the machine $M$ is in state $s$, sees the symbol $x$, prints the symbol $y$, goes to $d$ and finally transits to state $r$. Any $\langle s, x, y, d, r \rangle$ represents a computational event.

3) $\delta$ is the amplitude function that assigns to any computational event $\langle s, x, y, d, r \rangle$ a complex number. The number $|\delta(\langle s, x, y, d, r \rangle)|^2$ will represent the probability of the computational event $\langle s, x, y, d, r \rangle$.

$M$ evolves in time assuming different states. A configuration (or basic state) of $M$ is a triplet:

\[ \varphi = \langle n, s, w \rangle \]

where $n$ is a location of $M$ on the tape, $s$ is a memory state and $w$ is a word, describing the condition of the tape. Similarly to the classical case, a word $w$ can be described as function

\[ w : \mathbb{Z} \rightarrow I \]
with the usual restriction: for almost all $n \in \mathbb{Z}$: $w(n) = \#$. Hence, a word corresponds to a finite sequence of symbols labelled by the cells of the tape. Instead of $w(n)$ we will also write $w_n$.

Let $B$ be the set of all possible basic states $\varphi$ (which describe definite configurations of $M$).

4) $\mathcal{H}$ is a Hilbert space with orthonormal basis $B$.

The amplitude function $\delta$ determines a linear operator

$$U : \mathcal{H} \to \mathcal{H}.$$ 

It is sufficient to define $U$ on $B$: for any $\varphi \in B$, $U(\varphi)$ will generally be a superposition state:

$$\sum_i c_i \varphi_i.$$ 

Let $\varphi = \langle n, s, w \rangle$ and let $w_n$ represent the symbol printed in the cell $n$.

Let us consider the set of all computational events starting with the pair $\langle s, w_n \rangle$ (which is determined by $\varphi$):

$$\langle s, w_n, y, d, r \rangle$$

for all possible $y, d, r$.

For any $y, d, r$, we define the complex number $c_{y,d,r}$ and the vector $\varphi_{y,d,r}$ as follows:

$$c_{y,d,r} = \delta(\langle s, w_n, y, d, r \rangle)$$

$$\varphi_{y,d,r} = \langle n(d), r, w(y,n) \rangle.$$ 

Where:

$$n(d) = \begin{cases} 
  n + 1, & \text{if } d = R; \\
  n - 1, & \text{if } d = L,
\end{cases}$$

and:

$$w(y,n)_m = \begin{cases} 
  y, & \text{if } m = n; \\
  w_m, & \text{otherwise}.
\end{cases}$$

Now, let us define:

$$U(\varphi) = \sum_{y,d,r} c_{y,d,r} \varphi_{y,d,r}.$$ 

We assume the following restriction: $U$ must be a unitary operator. Obviously, such a restriction represents a constraint on the amplitude function $\delta$. 

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How can the action of our unitary operator $U$ be interpreted? Let $\psi_0 = \langle 0, s_0, w \rangle$ represent the element of $B$ that is the initial state, where the word $w$ corresponds to the input for $M$. The state $U(\psi_0)$ will represent the evolution of the initial state $\psi_0$ after one step. Generally $U^t(\psi_0)$ will represent the evolution of $\psi_0$ after $t$ steps (we might also say: at time $t$).

What does it mean that our machine $M$ halts at time $t$ with a certain probability value? In order to answer this question, let us first introduce the notion of final basic state.

**Definition 4.2** A final basic state is a vector $\varphi \in B$ s.t.:

$$\varphi = \langle n, s_f, w \rangle,$$

where $s_f$ is the final memory state.

We will denote by $B^F$ set of all final basic states.

**Definition 4.3** The probability that $M$ halts at time $t$ ($\text{Prob}^H(t)$) is defined as follows:

$$\text{Prob}^H(t) = \sum_{\varphi \in B^F} |(U^t(\psi_0), \varphi)|^2$$

(where $(U^t(\psi_0), \varphi)$ represents the inner product of $U^t(\psi_0)$ and $\varphi$).

> From an intuitive point of view, any sequence $

\langle \psi_0, \ldots, U^t(\psi_0) \rangle

$ can be regarded as a knowledge path (or an epistemic history) of $M$. Such a sequence naturally gives rise to a graph, where each branch has the form:

$$\langle \psi_0, \varphi_1, \ldots, \varphi_t \rangle$$

with $\varphi_i \in B$.

These basic sequences $\langle \psi_0, \varphi_1, \ldots, \varphi_t \rangle$ are determined as follows. For any $k$ ($0 \leq k \leq t$), consider

$$U^k(\psi_0) = \sum_i c_i \varphi_i \quad \text{(with } c_i \neq 0).$$

We require:

$$\varphi_k = \varphi_i, \quad \text{for a given } \varphi_i \text{ occurring in } \sum_i c_i \varphi_i.$$  

Since states correspond to unidimensional closed subspaces of $\mathcal{H}$, any knowledge path of $M$ represents a historical sequence (in the sense of Definition 2.1).

On this basis, one can distinguish at least two different notions of computation.
Definition 4.4 A **strong computation** of $M$ is a knowledge path

$$\langle \psi_0, ..., U^t(\psi_0) \rangle$$

where $M$ halts at time $t$ with probability 1.

Definition 4.5 A **weak computation** is a knowledge path

$$\langle \psi_0, ..., U^t(\psi_0) \rangle$$

where $M$ halts at $t$ with probability different from 0.

In this framework, one can naturally introduce some semantic notions (following a standard quantum logical style).

Definition 4.6 A basic state $\varphi$ **knows** (or **accepts**) a word $w$ ($\varphi \models w$) iff there is a location $n$ and a memory state $s$ such that

$$\varphi = \langle n, s, w \rangle$$

Definition 4.7 A state $\psi$ **possibly knows** (or **possibly accepts**) a word $w$ ($\psi \triangledown w$) iff

$$\psi = \sum c_i \varphi_i$$

with $c_i \neq 0$ and for at least one $\varphi_i$,

$$\varphi_i \models w.$$  

Definition 4.8 $M$ **possibly knows** a word $w$ at time $t$ ($M \triangledown_t w$) iff

$$U^t(\psi_0) \triangledown w.$$ 

Definition 4.9 $M$ **knows** a word $w$ at time $t$ ($M \models_t w$) iff

1) $M \triangledown_t w$;

2) $M$ halts at time $t$ with probability 1.

Definition 4.10 The **proposition** $C(w)$ of a word $w$ is the smallest closed subspace that includes

$$\{ \varphi \in B \mid \varphi \models w \}.$$ 

Suppose we equip our language with the logical connectives $\neg$ and $\land$ (where the atomic formulas can be identified with particular words). Then we can give the usual (quantum logical) semantic definitions:

Definition 4.11

$$\psi \triangledown \neg \alpha \iff \psi \in C(\alpha)^c;$$

$$\psi \triangledown \alpha \land \beta \iff \psi \in C(\alpha) \cap C(\beta).$$
Suppose $M \vdash_t \alpha$ and $M \models_t \alpha$ are defined as above. It may happen:

$$\exists t : M \models_t \neg \alpha$$

And

$$\exists t : M \models_t \alpha$$

In other words, the negation $\neg$ shows in this framework a weak paraconsistent behaviour (similarly to a negation by failure).

References


