The Game of Chaos

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Noam Nisan
Algorithms for Selfish Agents
Mechanism Design for Distributed Computing
Invited paper.

Elias Koutsoupias & Christos Papadimitriou

Game Theory for Controlling the Internet!
The Game of Chaos

Sorry: it is a French Card
Game of Chaos
Sorcery

Play head or tails against a
target opponent. The looser
of the game looses one life.
The winner of the game gains
one life, and may choose to
repeat the procedure. For every
repetition the ante in life is
doubled.

Magic; the Gathering

Customizable card game: build a deck using a very large collection
of available cards.
Both players start out with 20 lives.
Number of lives \( \leq 0 \) means you have lost the duel.
Move = playing land, casting a spell, combat, ....
Attack: summoning creatures, damaging spells, damaging effects
Defense: Blocking attacking creatures, protecting spells and effects
Spells require Mana obtained by tapping lands or activating other
Mana sources. Mana exists in 5 colors and a generic variant.
Spells exist in the same 5 colors or a generic variant (artifacts)

For almost every rule in the game there exists a card creating an
exception against it when successfully cast.....
The Players

URGAT

THORGRIM

Game Trees

Non Zero-Sum Game:
Payoffs explicitly designated at terminal node

Terminal node

Root

Thorgrim’s turn

Urgat’s turn

1 / -1
-3 / 2
1 / 4
5 / -7
3 / 1
2 / 0
-1 / 4
**Backward Induction**

At terminal nodes: Pay-off as explicitly given

At Thorgrim’s nodes: Pay-off inherited from Thorgrim’s optimal choice

At Urgat’s nodes: Pay-off inherited from Urgat’s optimal choice

For strictly competitive games this is the Max-Min rule

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**CHANCE MOVES**

- Chance moves controlled by another player (Nature) who is not interested in the result
- Nature is bound to choose his moves fairly with respect to commonly known probabilities
- Resulting outcomes for active players become lotteries
Lotteries

![Lottery Diagram]

Expectation:
\[
\frac{1}{2} \cdot -2 + \frac{1}{6} \cdot 12 + \frac{1}{3} \cdot 3 = 2
\]

Compound Lottery

![Compound Lottery Diagram]

In compound lotteries all drawings are assumed to be independent
Flipping a coin

**HEA D S**

1/2

h

1/2

1 / -1

**T A I L S**

1/2

l

1/2

1 / -1

Expectation

0 / 0

0 / 0

Thorgrim calls head or tails and Urgat flips the coin. Urgat’s move is irrelevant. Nature determines the outcome.

The Game Tree

Thorgrim and Urgat both start with 5 lives
WHY UTILITY FUNCTIONS?

- Backward Induction is based on preferences rather than numbers
- Numbers as a tool for expressing preferences works OK when chance moves are absent
- We like to compute expected pay-off at chance nodes.
- Expected pay-off is sensitive to scaling
- Comparing complex lotteries is non-trivial

Comparing Complex Lotteries
Allais Example

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??

0 1 0
$0M $1M $5M

0.01 0.89 0.10
$0M $1M $5M

0.89 0.11 0
$0M $1M $5M

0.9 0 0.1
$0M $1M $5M
Von Neumann-Morgenstern Utility

Rational Players may be assumed to maximize the expectation of Something.
Let’s call this Something Utility.

Works nice for 2-outcome Lotteries:
Something = chance of winning.

So let’s reduce the n-outcome Lotteries to 2-outcome Compound Lotteries:
Each intermediate outcome is “equivalent” to a suitable 2-outcome Lottery. The involved chance determines the Utility.

Utility Intermediate Outcome

If $p$ is large (almost 1): Lot-1 > Lot-3
For $p$ small (almost 0): Lot-1 < Lot-3
So for some intermediate $p$, say $q$: Lot-1 $\approx$ Lot-3

$q \approx$ Lot-3 whence $u(D) = q.b + (1-q).a$!
Utility Lottery = Expected Utility Outcomes

Lot-1
\[ p_1 \quad p_2 \quad p_n \]
\[ \alpha_1 \quad \alpha_2 \quad \alpha_n \]
\[ \approx \]
Lot-2
\[ p_1 \quad p_2 \quad p_n \]
\[ q_i \quad 1-q_i \]
\[ W \quad L \]
\[ \Sigma_{p,q_i} \]
\[ \approx \]
Lot-3
\[ \Sigma_{p,q_i} \quad 1-\Sigma_{p,q_i} \]
\[ W \quad L \]
\[ u(W) = 1, \ u(L) = 0, \ u(\alpha_i) = q_i \]
\[ \Sigma_{p,q_i} = u(\text{Lot-3}) = \Sigma_{p_i}(\alpha_i) = E_{\text{Lot-1}} u(\text{outcome}) \]

Game of Chaos

Structure of the game tree independent of the choice of the utilities.

\[ u_{r,1}: \ u_{r,1}(n) = n \]
\[ u_{r,2}: \ u_{r,2}(n) = \begin{cases} n \geq v_{opp} & \text{then 1} \\ \text{iff} \ n \leq -v_{self} & \text{then -1} \\ \text{else} & 0 \end{cases} \]

\[ u_{u,1} \]
\[ u_{u,1}(n) = -n \]
\[ u_{u,2}: \ u_{u,2}(n) = \begin{cases} n \geq v_{self} & \text{then -1} \\ \text{iff} \ n \leq -v_{opp} & \text{then 1} \\ \text{else} & 0 \end{cases} \]
Linear Utilities

Thorgrim and Urgat both start with 5 lives

Both Thorgrim and Urgat use utility $u_1$

Go for the Kill!

Thorgrim and Urgat both start with 5 lives

Both Thorgrim and Urgat use utility $u_2$
### Unequal Start

**Thorgrim:** 6 lives  
**Urgat:** 4 lives  
Utilities used: $u_2$

![Game Tree Diagram](Image)

### Thorgrim's last stand

**Thorgrim:** 1 life  
**Urgat:** 6 lives

![Game Tree Diagram](Image)

Utilities: Thorgrim uses $u_3$: $u_3(n) = \begin{cases} n \geq v_{\text{opp}} & \text{then 1} \\ \text{else} & 0 \end{cases}$  
Urgat uses $u_2$