Modal Logics of Submaximal and Nodec Spaces

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Dedicated to Professor Dick De Jongh on his 65th birthday

Abstract

In this note we investigate modal logics of submaximal and nodec spaces when the modal diamond is interpreted as the closure operator of a topological space. We axiomatize the modal logic of nodec spaces. We show that the modal logic of submaximal spaces is a proper extension of the modal logic of nodec spaces, axiomatize it, and prove that it coincides with the modal logics of door spaces and I-spaces. We also show that the modal logic of maximal spaces is a proper extension of the modal logic of submaximal spaces, axiomatize it, and prove that it coincides with the modal logic of perfectly disconnected spaces.

1 Introduction

In [16] McKinsey and Tarski suggested to interpret the modal diamond as the closure operator of a topological space, and showed that under such interpretation, the basic modal logic of all topological spaces is **S4**. One of the main results of [16] states that **S4** is complete with respect to any metric separable dense-in-itself space. In particular, **S4** is complete with respect to the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, or the Cantor space $\mathcal{C}$.

To mention a few other topological completeness results, recall that a topological space X is called *extremally disconnected* if the closure of every open subset of X is open; it is called *scattered* if every non-empty subspace of X contains an isolated point; X is called *weakly scattered* if the set of isolated points of X is dense in X.[1] We call X a *McKinsey space* if the set of dense subsets of X forms a filter. Also, X is called *irresolvable* if X is not the union of two disjoint dense subsets of X, and it is called *hereditarily irresolvable (HI)* if every subspace of X is irresolvable. Then it is known that $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$ is the logic of all extremally disconnected spaces [12], and that $\mathbf{S4.1} = \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p$ is the logic of all McKinsey spaces, which coincides with the logic all weakly scattered spaces [12, 5]. Moreover, $\mathbf{S4.Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$ is the logic of all HI spaces, and it coincides with the logic of all scattered spaces, the logic of all ordinal spaces, or the logic of any ordinal $\alpha \geq \omega^\omega$ [9, 1, 5].

The aim of this paper is to add to the abovementioned topological completeness results. In particular, we will consider the class of nodec spaces, its subclass

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[1]Weakly scattered spaces are sometimes called α-scattered (see, e.g., [18]).

of submaximal spaces, and such subclasses of submaximal spaces as the classes of door spaces, I-spaces, maximal spaces, and perfectly disconnected spaces. We axiomatize the modal logic of nodec spaces; show that the modal logic of submaximal spaces is its proper normal extension, axiomatize it, and prove that it coincides with the modal logics of door spaces and I-spaces. We also show that the modal logic of maximal spaces is a proper normal extension of the modal logic of submaximal spaces, axiomatize it, and prove that it coincides with the modal logic of perfectly disconnected spaces.

2 Submaximal and nodec spaces

We recall that a topological space X is called *submaximal* if every dense subset of X is open, and that X is called *nodec* if every nowhere dense subset of X is closed. Different equivalent conditions for a space to be submaximal are given in [3, Theorem 1.2], and the ones for a space to be nodec in [6, Fact 1.14] and [17, Corollary to Proposition 4].[2] In particular, they imply that every submaximal space is nodec. The converse is not true: any trivial topology on a set with more than two elements is nodec, but not submaximal. This example shows that there exist nodec spaces that are not T_0. On the other hand, it is known (see, e.g., [5, Remark 2.6]) that every submaximal space is T_0.[3]

We point out that in Theorem 1.2 of [3], the conditions (d) and (f) require that the space X under consideration be T_1. We remove this restriction by adding an extra condition to both (d) and (f). Throughout I, C, and d will denote the interior, closure, and derived set operators, respectively. The complement of a set A will be denoted by A^c.

Theorem 2.1 *The following conditions are equivalent:*

1. *X is submaximal.*
2. *$CA - A$ is closed for each $A \subseteq X$.*
3. *For each $A \subseteq X$, if $IA = \emptyset$, then A is closed and discrete.*
4. *$CA - A$ is closed and discrete for each $A \subseteq X$.*

Proof. (1) $\Leftrightarrow$ (2) is the equivalence $(g) \Leftrightarrow (e)$ of [3, Theorem 1.2].

(2) $\Rightarrow$ (3) Suppose $IA = \emptyset$. Then

$$A^c = A^c \cup IA = A^c \cup (C(A^c))^c = (A \cap C(A^c))^c = (C(A^c) - (A^c))^c$$

is open since $C(A^c) - (A^c)$ is closed. So A is closed. Thus, $dA \subseteq A$. We show that $dA = \emptyset$. Let $x \in A$. Since $IA = \emptyset$, we also have that $I(A - \{x\}) = \emptyset$. Therefore, $A - \{x\}$ is closed, and so $\{x\} \cup A^c$ is open. But then there is an open

[2] We point out that in [17] nodec spaces are called α-topologies.

[3] In fact, as follows from Corollary 2.4 below, every submaximal space is T_D.

neighborhood $U_x = \{x\} \cup A^c$ of x such that $U_x \cap (A - \{x\}) = \emptyset$. Thus, $x \notin dA$. It follows that $dA = \emptyset$. Therefore, A is closed and discrete.

(3) $\Rightarrow$ (4) Since

$$I(CA - A) = ICA \cap I(A^c) = ICA \cap (CA)^c = \emptyset$$

we have that $CA - A$ is closed and discrete.

(4) $\Rightarrow$ (2) Obvious. $\dashv$

To this end, we call closed and discrete sets simply *clods*. We recall that the *Hausdorff residue* $\rho(A)$ of a subset A of a space X is defined as $\rho(A) = A \cap C(CA - A)$.

Lemma 2.2 *For $A \subseteq X$ the following hold.*

1. *$CA - A$ is closed iff $\rho(A) = \emptyset$.*
2. *$CA - A$ is clod iff $d(dA - A) = \emptyset$.*

Proof. (1) If $CA - A$ is closed, then

$$\rho(A) = A \cap C(CA - A) = A \cap (CA - A) = \emptyset$$

Conversely, if $\rho(A) = \emptyset$, then $C(CA - A) \subseteq A^c$. We also have that $C(CA - A) \subseteq CA$. Therefore, $C(CA - A) \subseteq CA \cap A^c = CA - A$. So $CA - A$ is closed.

(2) Since $CA - A = (A \cup dA) - A = dA - A$, we have that $CA - A$ is clod iff $dA - A$ is clod iff $d(dA - A) = \emptyset$. $\dashv$

Corollary 2.3 *The following two conditions are equivalent to the four conditions of Theorem 2.1:*

5. *$\rho(A) = \emptyset$ for each $A \subseteq X$.*
6. *$d(dA - A) = \emptyset$ for each $A \subseteq X$.*

Proof. It follows immediately from Theorem 2.1 and Lemma 2.2. $\dashv$

We recall that a space X satisfies the T_D *separation axiom* or is a T_D*-space* if every point in X is the intersection of an open and a closed subset of X. Equivalently, X is T_D iff $ddA \subseteq dA$ for each $A \subseteq X$. It is well known that the T_D separation axiom is strictly in between the T_0 and T_1 separation axioms.

Corollary 2.4

1. *If X is submaximal, then X is HI.*
2. *If X is submaximal, then X is T_D.*

Proof. (1) It follows from [5, Theorem 2.4] that X is HI iff $\rho(A) \subsetneq A$ for each nonempty subset A of X. Now if X is submaximal and A is a nonempty subset of X, then $\rho(A) = \emptyset \subsetneq A$. So X is HI.

(2) Every HI space is T_D. To see this, let $x \in X$. We need to show that x is isolated in $C(x)$. If not, then $C(x) = C(C(x) - \{x\})$. Therefore, $\{x\}$ and $C(x) - \{x\}$ are disjoint dense subsets of $C(x)$, implying that $C(x)$ is reducible. Now apply (1).[4] $\dashv$

The converse of Corollary 2.4 is not true: already any ordinal $\alpha \geq \omega^2 + 1$ is not a submaximal space.

Theorem 2.5 *The following conditions are equivalent:*

1. *X is nodec.*
2. *Each nowhere dense subset of X is clod.*
3. *For each $A \subseteq X$, if $A \subseteq ICIA$, then A is open.*
4. *For each $A \subseteq X$, if $CICA \subseteq A$, then A is closed.*
5. *$dA \subseteq CICA$ for each $A \subseteq X$.*
6. *$CA = A \cup CICA$ for each $A \subseteq X$.*
7. *$IA = A \cap ICIA$ for each $A \subseteq X$.*

Proof. For (1)⇔(2) see [6, Fact 1.14], and for (1)⇔(3) see [17, Corollary to Proposition 4].

(3)⇔(4) is obvious.

(2)⇒(5) Let $A \subseteq X$. Since

$$\begin{aligned} IC(A - ICA) &= IC(A \cap (ICA)^c) = IC(A \cap CI(A^c)) \subseteq I(CA \cap CI(A^c)) \\ &= ICA \cap ICI(A^c) = ICA \cap (CICA)^c = ICA - CICA = \emptyset \end{aligned}$$

we have that $A - ICA$ is nowhere dense. Therefore, $A - ICA$ is clod. Thus, $d(A - ICA) = \emptyset$, and as $dICA \subseteq CICA$ and $dA - dB \subseteq d(A - B)$, we have that

$$dA - CICA \subseteq dA - dICA \subseteq d(A - ICA) = \emptyset$$

It follows that $dA \subseteq CICA$.

(5)⇒(6) As $A, CICA \subseteq CA$, we have that $A \cup CICA \subseteq CA$. Conversely, $CA = A \cup dA \subseteq A \cup CICA$. Thus, the equality.

(6)⇔(7) is obvious.

(6)⇒(1) If $N \subseteq X$ is nowhere dense, then $CN = N \cup CICN = N$. So N is closed. $\dashv$

We recall that a space X is said to be a *door* space if every subset of X is either open or closed. It is obvious that every door space is submaximal. The converse however is not true: the spaces in [5, Proposition 3.4], where the

[4] We point out that if X is submaximal, then every point in X is in fact either open or closed. To see this, if $x \in X$ is not isolated, then $\{x\}^c$ is dense, so open, and so $\{x\}$ is closed.

original space is not a door space, are submaximal but not door. For more examples see Lemma 3.1 below.

We also recall that a space X is called an *I-space* if $ddX = \emptyset$. It is pointed out in [3] that for a space X the following three conditions are equivalent: (i) X is an I-space; (ii) X is nodec and (weakly) scattered; (iii) X is submaximal and (weakly) scattered. Examples of I-spaces that are not door are the ordinals $\alpha \in [\omega 2 + 1, \omega^2]$. For examples of door spaces that are not I-spaces, recall that a space X is called *filtral* if the set $\tau - \{\emptyset\}$ of nonempty open subsets of X is a filter. Let X be an infinite filtral space, where $\tau - \{\emptyset\}$ is a free ultrafilter. Then X is a dense-in-itself door space [7], hence is not an I-space.

A space X is called *maximal* if every open subset of X is infinite and any strictly finer topology on X contains a finite open set. It is known (see, e.g., [13, Theorem 24]) that every maximal space is submaximal. Since maximal spaces are dense-in-itself and I-spaces are (weakly) scattered, the two classes have the empty intersection. The filtral spaces, where the filter $\tau - \{\emptyset\}$ is a principal ultrafilter, serve as examples of door spaces that are not maximal. For examples of maximal spaces that are not door, we note that it was shown in [13, Theorem 13] (see also [6, Theorem 1.2(b)]) that there exist Hausdorff maximal spaces. We point out that none of them can be door.

Closely related to the notion of maximality is the notion of perfectly disconnected spaces from [6]. We recall that a space X is called *perfectly disconnected* if X is T_0 and disjoint subsets of X have no common limit points. Equivalently, X is perfectly disconnected iff X is T_0 and $dA \cap d(A^c) = \emptyset$ for each $A \subseteq X$. It is shown in [6, Theorem 2.2] that if X is dense-in-itself, then X is maximal iff X is perfectly disconnected. It follows that maximal spaces are perfectly disconnected. The Sierpinski space serves as an example of a perfectly disconnected space that is not maximal. Since the class of maximal spaces does not intersect with the class of I-spaces and since there exist maximal spaces that are not door, it follows that there exist perfectly disconnected spaces that are neither door nor I-spaces. The ordinal $\omega + 1$ serves as an example of a door and an I-space that is not perfectly disconnected. For more examples see Lemmas 3.1 and 3.5 below.

Proposition 2.6 *If X is perfectly disconnected, then X is submaximal.*

Proof. We first show that X is a T_D-space. Suppose not. Then there exists $A \subseteq X$ such that $ddA \not\subseteq dA$. Therefore, there is $x \in ddA$ such that $x \notin dA$. The latter implies that there is an open neighborhood U_x of x such that $U_x \cap (A - \{x\}) = \emptyset$. Consequently, $U_x \subseteq A^c \cup \{x\}$. From $x \in ddA$ it follows that $U_x \cap (dA - \{x\}) \neq \emptyset$. So there is $y \neq x$ such that $y \in U_x \cap dA$. We show that $y \in dA \cap d(A^c)$. That $y \in dA$ follows from the selection of y. To show that $y \in d(A^c)$, we first show that $y \in d(x)$. Let V be an open neighborhood of y. It is sufficient to show that $x \in V$. We set $U = V \cap U_x$. Since $y \in dA$, we have that $U \cap (A - \{y\}) \neq \emptyset$. ¿From $U \subseteq U_x \subseteq A^c \cup \{x\}$ it follows that $U \cap (A - \{y\}) = \{x\}$. Thus, $x \in U \subseteq V$, implying that $y \in d(x)$. Now since X is T_0 and $y \in d(x)$, there must exist an open neighborhood V_x of x such that $y \notin V_x$. Let V be an open neighborhood of y. We set $O = V \cap U_x \cap V_x$. Since $x \in ddA$ and O is an open neighborhood of x, we have that $O \cap (dA - \{x\}) \neq \emptyset$. So there exists $z \neq x$ such that $z \in O \cap dA$. Since $z \neq x$ and $z \in O \subseteq V \cap U_x \cap V_x \subseteq U_x \subseteq A^c \cup \{x\}$, we have that $z \in A^c$. Clearly $z \neq y$ as $y \notin V_x$. Therefore, $z \in V \cap (A^c - \{y\})$. It follows that for any open neighborhood V of y we have $V \cap (A^c - \{y\}) \neq \emptyset$. This implies that $y \in d(A^c)$. Thus, $y \in dA \cap d(A^c)$, which is a contradiction because X was assumed to be perfectly disconnected. Therefore, $ddA \subseteq dA$ for each $A \subseteq X$, and so X is T_D.

To complete the proof, suppose X is not submaximal. Then by Corollary 2.3 there exists $A \subseteq X$ such that $d(dA - A) \neq \emptyset$. Since d preserves $\subseteq$ and X is T_D, we obtain that $d(dA - A) = d(dA \cap A^c) \subseteq ddA \subseteq dA$ and $d(dA - A) = d(dA \cap A^c) \subseteq d(A^c)$. Therefore, $\emptyset \neq d(dA - A) \subseteq dA \cap d(A^c)$, and so there exists $A \subseteq X$ such that $dA \cap d(A^c) \neq \emptyset$. Thus, X is not perfectly disconnected, which is a contradiction. $\dashv$

Therefore, we obtain the following relationship between the above six classes of spaces:

$$\begin{array}{ccccccc} & & & & \textbf{Door} & & \\ & & & & \cap & & \\ \textbf{Maximal} & \subset & \textbf{Perfectly disconnected} & \subset & \textbf{Submaximal} & \subset & \textbf{Nodec} \\ & & & & \cup & & \\ & & & & \textbf{I-spaces} & & \end{array}$$

We conclude this section by a characterization of perfectly disconnected spaces in terms of extremally disconnected spaces, which, in view of [6, Theorem 2.2], can be thought of as a generalization of the characterization of maximal spaces given in [14, 15].

Theorem 2.7 *A space X is perfectly disconnected iff X is submaximal and extremally disconnected.*

Proof. That perfectly disconnected spaces are submaximal follows from Proposition 2.6; that they are extremally disconnected follows from [8, Theorem 6.2.26]. Conversely, suppose that X is submaximal and extremally disconnected. Because X is submaximal, X is T_0. Moreover, for $A \subseteq X$, we have that $d(A - IA) = d(A \cap (IA)^c) = d(A \cap C(A^c)) = d(C(A^c) - (A^c)) = d(d(A^c) - (A^c)) = \emptyset$ by Corollary 2.3. Therefore, $dA = d((A - IA) \cup IA) = d(A - IA) \cup dIA = dIA$. Similarly, $d(A^c - I(A^c)) = \emptyset$ and so $d(A^c) = dI(A^c)$. Thus, $dA \cap d(A^c) = dIA \cap dI(A^c) = \emptyset$ as X is extremally disconnected. It follows that X is perfectly disconnected. $\dashv$

3 Modal logics of submaximal and nodec spaces

We recall that **S4** is the least set of formulas of the basic modal language $\mathcal{L}$ containing the axioms (i) $\Box(p \to q) \to (\Box p \to \Box q)$, (ii) $\Box p \to p$, and (iii) $\Box p \to \Box\Box p$, and closed under modus ponens, substitution, and necessitation $(\varphi/\Box\varphi)$. A topological model is a pair $\langle X, \nu \rangle$, where X is a topological space and ν is a valuation, assigning to each propositional variable of $\mathcal{L}$ a subset of X. The connectives $\vee, \wedge$, and $\neg$ are interpreted in $\langle X, \nu \rangle$ as the set-theoretical union, intersection, and complement; and the modal operators $\Box$ and $\Diamond$ are interpreted as the interior and closure operators of X. As usual, $x \models \varphi$ denotes that φ is satisfied in $x \in X$; we say that φ is *true* in $\langle X, \nu \rangle$ if $x \models \varphi$ for each $x \in X$; that φ is *valid in* X if φ is true in $\langle X, \nu \rangle$ for each valuation ν; and that φ is *valid in a class* $\mathcal{K}$ of topological spaces if φ is valid in each member of $\mathcal{K}$.

For a class $\mathcal{K}$ of spaces, let $L(\mathcal{K})$ denote the set of formulas of $\mathcal{L}$ that are valid in $\mathcal{K}$. It is easy to verify that $L(\mathcal{K})$ is a normal extension of **S4**. We call $L(\mathcal{K})$ the *modal logic of* $\mathcal{K}$. Let $\mathcal{T}$ denote the class of all topological spaces; $\mathcal{ED}$ the class of all extremally disconnected spaces; $\mathcal{MK}$ the class of all McKinsey spaces; $\mathcal{SCAT}$ the class of all scattered spaces; $\mathcal{WSCAT}$ the class of all weakly scattered spaces; $\mathcal{ORD}$ the class of all ordinal spaces; and $\mathcal{HI}$ the class of all hereditarily irresolvable spaces. Then the completeness results for modal logics mentioned in the introduction can be stated as follows:

1. $\mathbf{S4} = L(\mathcal{T}) = L(\mathbb{R}) = L(\mathbb{Q}) = L(\mathcal{C})$.
2. $\mathbf{S4.2} = L(\mathcal{ED})$.
3. $\mathbf{S4.1} = L(\mathcal{MK}) = L(\mathcal{WSCAT})$.
4. $\mathbf{S4.Grz} = L(\mathcal{HI}) = L(\mathcal{SCAT}) = L(\mathcal{ORD}) = L(\omega^\omega)$.

Our goal in this section is to add to the above completeness results and axiomatize modal logics of the six classes of spaces described in Section 2.

We recall that a space X is called *Alexandroff* if the intersection of any family of open subsets of X is again open. Equivalently, X is Alexandroff iff every $x \in X$ has a least open neighborhood. It is well known that Alexandroff spaces correspond to **S4**-frames (see, e.g., [2]). Indeed, we recall that a **S4**-frame is a pair $\mathfrak{F} = \langle X, R \rangle$, where $R \subseteq X^2$ is reflexive and transitive. For a given $\mathfrak{F}$, a subset A of X is called an *upset* of $\mathfrak{F}$ if $x \in A$ and xRy imply $y \in A$. Dually, A is called a *downset* if $x \in A$ and yRx imply $y \in A$. The topology on X is defined by declaring the upsets of $\mathfrak{F}$ to be open. Then the downsets of $\mathfrak{F}$ turn out to be closed, and it is routine to verify that the obtained space is Alexandroff, that a least neighborhood of $x \in X$ is $R(x) = \{y \in X : xRy\}$, that the closure of a set $A \subseteq X$ is

$$C_R(A) = R^{-1}(A) = \{x \in X : \exists y \in A \text{ with } xRy\}$$

and that the interior of A is

$$I_R(A) = (R^{-1}(A^c))^c = \{x \in X : (\forall y \in X)(xRy \Rightarrow y \in A)\}$$

For a topological space X, define the *specialization order* on X by setting xRy iff $x \in C(y)$. Then it is easy to check that the specialization order is reflexive and transitive,[5] and that the upsets of $\langle X, R\rangle$ are exactly the opens of X iff X is Alexandroff. These observations immediately imply that there is a 1-1 correspondence between Alexandroff spaces and **S4**-frames, and hence every Kripke complete normal extension of **S4** is also topologically complete.

Let $\mathcal{N}$ denote the class of all nodec spaces; $\mathcal{S}$ the class of all submaximal spaces; $\mathcal{D}$ the class of all door spaces; $\mathcal{I}$ the class of all I-spaces; $\mathcal{PD}$ the class of all perfectly disconnected spaces; and $\mathcal{M}$ the class of all maximal spaces. It follows from Section 2 that $\mathcal{M} \subsetneqq \mathcal{PD} \subsetneqq \mathcal{S} \subsetneqq \mathcal{N}$, that $\mathcal{D}, \mathcal{I} \subsetneqq \mathcal{S}$, that $\mathcal{M} \cap \mathcal{I} = \emptyset$, and that $\mathcal{M}, \mathcal{D}$; $\mathcal{D}, \mathcal{I}$; $\mathcal{PD}, \mathcal{D}$ and $\mathcal{PD}, \mathcal{I}$ are pairwise incomparable. The rest of this section is dedicated to showing that

1. $L(\mathcal{N}) = \mathbf{S4.Zem} = \mathbf{S4} + \Box\Diamond\Box p \to (p \to \Box p)$,
2. $L(\mathcal{S}) = L(\mathcal{I}) = L(\mathcal{D}) = \mathbf{S4} + p \to \Box(\Diamond p \to p)$,
3. $L(\mathcal{PD}) = L(\mathcal{M}) = \mathbf{S4.2} + p \to \Box(\Diamond p \to p)$.

Let X be an Alexandroff space and R be the specialization order on X. Then the opens of X are exactly the upsets of $\langle X, R\rangle$. We call $\langle X, R\rangle$ *rooted* if there exists $r \in X$ such that rRx for each $x \in X$. If this is the case, r is called a *root* of $\langle X, R\rangle$. We call $x \in X$ *maximal* if xRy implies $x = y$, and *quasi-maximal* if xRy implies yRx; similarly, $x \in X$ is called *minimal* if yRx implies $y = x$, and *quasi-minimal* if yRx implies xRy. Let $maxX$ and $qmaxX$ denote the sets of maximal and quasi-maximal points, and $minX$ and $qminX$ the sets of minimal and quasi-minimal points of X. If R is a partial order, it is obvious that $maxX = qmaxX$ and $minX = qminX$. We call $Y \subseteq X$ a *quasi-chain* if for every $x, y \in Y$ we have that xRy or yRx. If in addition xRy implies $y\not R x$, then Y is called a *chain*. Again the two notions coincide if R is a partial order. A chain Y of X is said to be of *length* n if it consists of n elements. We say that $\langle X, R\rangle$ is of *depth* n if there exists a chain in X of length n and every other chain in X is of length $\leq n$.

Lemma 3.1 *Let X be an Alexandroff space with the specialization order R.*

1. *X is nodec iff $\langle X, R\rangle$ is of depth ≤ 2 and $qminX - qmaxX \subseteq minX$.*
2. *X is submaximal iff X is an I-space iff $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2.*
3. *X is door iff $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2 such that either $maxX - minX$ or $minX - maxX$ consists of just one point.*

[5] It is a partial order iff X is T_0.

Proof. (1) Suppose X is nodec and there is a chain $Y \subseteq X$ of length > 2. Let $xRyRz$ be three distinct elements from Y. Then $z\not{R}y$ and $y\not{R}x$. So $z \notin R^{-1}(y)$, and so $\{y\}$ is nowhere dense. However, it is not a downset as $x \notin R^{-1}(y)$, and hence $\{y\}$ is not closed, contradicting to X being nodec. Therefore, $\langle X, R\rangle$ is of depth ≤ 2. Similarly, if there exist $x, y \in qminX - qmaxX$ such that xRy and yRx, then $\{y\}$ is nowhere dense but not closed, which is again a contradiction. Thus, $qminX - qmaxX \subseteq minX$. Conversely, suppose $\langle X, R\rangle$ is of depth ≤ 2 and $qminX - qmaxX \subseteq minX$. Then $N \subseteq X$ is nowhere dense iff $N \cap qmaxX = \emptyset$. Therefore, if N is nowhere dense, then $N \subseteq minX$, which implies that N is a downset, hence closed.

(2) If X is an I-space, then X is submaximal. If X is submaximal, then X is nodec, so (1) implies that $\langle X, R\rangle$ is of depth ≤ 2. Also, since submaximal spaces are T_0, $\langle X, R\rangle$ is a partially ordered set. Suppose $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2. Since $maxX$ is the set of isolated points of X, we have that $ddX = d(X - maxX) = \emptyset$. Therefore, X is an I-space.

(3) Suppose X is door. Then X is submaximal and (2) implies that $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2. If both $maxX - minX$ and $minX - maxX$ consist of at least two points, then either there exist $x \in maxX - minX$ and $y \in minX - maxX$ such that $y\not{R}x$ or all points in $minX - maxX$ are R-related to all points in $maxX - minX$. In either case, $\{x, y\}$ is neither an upset nor a downset. Hence, $\{x, y\}$ is neither open nor closed, which contradicts to X being door. Conversely, if $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2 such that either $maxX - minX$ or $minX - maxX$ consists of at most one point, then as $X = maxX \cup minX$, it follows that X is door. $\dashv$

As an immediate consequence we obtain the following.

Corollary 3.2 *If X is an Alexandroff space with the specialization order R such that $\langle X, R\rangle$ is rooted, then the following conditions are equivalent.*

1. *X is submaximal.*
2. *X is an I-space.*
3. *X is door.*
4. *$\langle X, R\rangle$ is a partially ordered set of depth ≤ 2.*

Theorem 3.3 $L(\mathcal{S}) = L(\mathcal{D}) = L(\mathcal{I}) = \mathbf{S4} + p \to \Box(\Diamond p \to p)$.

Proof. Since $\mathcal{D}, \mathcal{I} \subseteq \mathcal{S}$, we have that $L(\mathcal{S}) \subseteq L(\mathcal{D}), L(\mathcal{I})$. We show that $p \to \Box(\Diamond p \to p)$ is valid in $\mathcal{S}$. Suppose X is a submaximal space and ν is a valuation on X. Denoting $\nu(p)$ by A and using the fact that $\rho(A) = \emptyset$ (see Corollary 2.3), we obtain that

$$\nu(p \to \Box(\Diamond p \to p)) = A^c \cup I((CA)^c \cup A) = A^c \cup (C(CA \cap A^c))^c =$$
$$(A \cap C(CA - A))^c = (\rho(A))^c = X$$

Therefore, $p \to \Box(\Diamond p \to p)$ is valid in every submaximal space, hence it is valid in $\mathcal{S}$. It follows that $\mathbf{S4} + p \to \Box(\Diamond p \to p) \subseteq L(\mathcal{S}) \subseteq L(\mathcal{D}), L(\mathcal{I})$. Now since $\mathbf{S4} + p \to \Box(\Diamond p \to p)$ is complete with respect to all finite rooted partial orders of depth 2 (see, e.g., [19, 11]), it follows from Corollary 3.2 that $\mathbf{S4} + p \to \Box(\Diamond p \to p) = L(\mathcal{D}) = L(\mathcal{I}) = L(\mathcal{S})$. $\dashv$

Theorem 3.4 $L(\mathcal{N}) = \mathbf{S4.Zem}$.

Proof. Suppose X is a nodec space and ν is a valuation on X. If we denote $\nu(p)$ by A, and use the fact that $IA = A \cap ICIA$ (see Theorem 2.5), then

$$\begin{aligned}\nu(\Box\Diamond\Box p \to (p \to \Box p)) = (ICIA)^c \cup A^c \cup IA = \\ (ICIA \cap A)^c \cup IA = (IA)^c \cup IA = X\end{aligned}$$

Therefore, $\Box\Diamond\Box p \to (p \to \Box p)$ is valid in every nodec space. It follows that $\mathbf{S4.Zem} \subseteq L(\mathcal{N})$. To show the converse, recall from [19, Theorem 7.5] that $\mathbf{S4.Zem}$ is complete with respect to all finite rooted frames of depth 2 with a unique root. Since by Lemma 3.1(1) these are nodec spaces, we obtain that $L(\mathcal{N}) \subseteq \mathbf{S4.Zem}$, thus the equality. $\dashv$

It follows that the modal logic of nodec spaces is $\mathbf{S4.Zem}$, and that the modal logic of submaximal spaces is $\mathbf{S4} + p \to \Box(\Diamond p \to p)$, which is one of the five pre-tabular extensions of $\mathbf{S4}$ described in [11]. Moreover, the latter coincides with the modal logics of door spaces and I-spaces, and is a proper normal extension of the former.

Lemma 3.5 *Let X be an Alexandroff space with the specialization order R.*

1. *X is perfectly disconnected iff $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2 such that $(\forall x, y, z \in X)(xRy \,\&\, xRz) \Rightarrow (\exists u \in X)(yRu \,\&\, zRu)$.*

2. *X is not maximal.*

Proof. (1) We recall that an Alexandroff space is extremally disconnected iff $(\forall x, y, z \in X)(xRy \,\&\, xRz) \Rightarrow (\exists u \in X)(yRu \,\&\, zRu)$ [12, Theorem 1.3.3]. Now using Theorem 2.7 and Lemma 3.1 we obtain that X is perfectly disconnected iff X is submaximal and extremally disconnected iff $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2 such that $(\forall x, y, z \in X)(xRy \,\&\, xRz) \Rightarrow (\exists u \in X)(yRu \,\&\, zRu)$.

(2) Suppose X is a maximal Alexandroff space. Then X is submaximal. So $\langle X, R\rangle$ is a partially ordered set of depth ≤ 2. Therefore, $maxX \neq \emptyset$. Thus, X has isolated points, contradicting to maximality of X. $\dashv$

We are in a position now to show that the modal logics of perfectly disconnected and maximal spaces coincide and are equal to $\mathbf{S4.2}{+}p \to \Box(\Diamond p \to p)$. We point out that since the two element chain is the only frame among the rooted frames of depth 2 that validate $\Diamond\Box p \to \Box\Diamond p$, and since $\mathbf{S4.2} + p \to \Box(\Diamond p \to p)$ is tabular, it is the logic of the two element chain.

Theorem 3.6 $L(\mathcal{PD}) = L(\mathcal{M}) = \mathbf{S4.2} + p \to \Box(\Diamond p \to p)$

Proof. As $\mathcal{M} \subseteq \mathcal{PD} \subseteq \mathcal{S}$, we have that $\mathbf{S4} + p \to \Box(\Diamond p \to p) = L(\mathcal{S}) \subseteq L(\mathcal{PD}) \subseteq L(\mathcal{M})$. Since $\Diamond\Box p \to \Box\Diamond p$ is valid in X iff X is extremally disconnected and since perfectly disconnected spaces are extremally disconnected, $\mathbf{S4.2} + p \to \Box(\Diamond p \to p) \subseteq L(\mathcal{PD})$. Moreover, as $\mathbf{S4.2} + p \to \Box(\Diamond p \to p)$ is the logic of the two element chain, i.e. the logic of the Sierpinski space, which is perfectly disconnected, we have $\mathbf{S4.2} + p \to \Box(\Diamond p \to p) = L(\mathcal{PD}) \subseteq L(\mathcal{M})$.

To show that $L(\mathcal{M}) = \mathbf{S4.2} + p \to \Box(\Diamond p \to p)$, it is sufficient to show that the Sierpinski space is a continuous and open image of any maximal space, and recall from [4, 12] that if Y is a continuous and open image of X, then $L(X) \subseteq L(Y)$. Let X be a maximal space. To construct a continuous and open map from X onto the Sierpinski space $S = \{u, v\}$, where $\{u\}$ is open and $\{v\}$ is closed, pick any $x \in X$ and set

$$f(y) = \begin{cases} v, & \text{if } y = x \\ u, & \text{otherwise} \end{cases}$$

It follows from the definition of f that it is a well-defined onto map. Since maximal spaces are T_1 and dense-in-itself, it is immediate that f is continuous and open. Therefore, $L(\mathcal{M}) = L(\mathcal{PD}) = \mathbf{S4.2} + p \to \Box(\Diamond p \to p)$. ⊣

Remark 3.7 Since both $\mathbf{S4.Zem}$ and $\mathbf{S4} + p \to \Box(\Diamond p \to p)$ have the same superintuitionistic companion, viz. the superintuitionistic logic of all finite rooted partially ordered sets of depth 2, the superintuitionistic logics of nodec, submaximal, door, and I-spaces coincide, and can be axiomatized by adding to the intuitionistic propositional logic **Int** the formula $q \vee (q \to (p \vee \neg p))$. The obtained logic is the least logic of the second slice of Hosoi, and is one of the three pre-tabular superintuitionistic logics.

Similarly, the superintuitionistic logics of perfectly disconnected and maximal spaces coincide with the superintuitionistic logic of the two element chain, and can be axiomatized by adding to **Int** the formulas $q \vee (q \to (p \vee \neg p))$ and $(p \to q) \vee (q \to p)$. This logic is the greatest logic of the second slice of Hosoi.

4 Further work

In the appendix to [16] McKinsey and Tarski suggested another interpretation of $\Diamond$; that is as the derived set operator d. For this interpretation, the basic modal logic of all topological spaces becomes **wK4** [10]. Since the derived set operator has more expressive power than the closure operator, logics over **wK4** can express such topological properties as being a T_D-space [10], a dense-in-itself space [20], or a scattered space [9]—the properties that logics over **S4** are not capable of expressing. Moreover, logics over **wK4** distinguish between $\mathbb{Q}$ and $\mathbb{R}$, as well as between $\mathbb{R}$ and higher dimensional Euclidean spaces [20]. None of this is distinguishable by logics over **S4**. In addition to this, we can show that logics over **wK4** are capable of distinguishing between all the six classes of topological spaces considered in this paper. However, these results, together with the finite d-axiomatization of these classes of spaces, will appear in the full version of this paper, which will be published elsewhere.

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