Smart labels

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Abstract

The notion of a critical successor [dJV90] has been central to all modal completeness proofs in interpretability logics. In this paper we shall work with an alternative notion, that of an assuring successor. As we shall see, this makes life a lot easier. After a general treatment of assuringness, we shall apply it to obtain a completeness results for **IL**W, a result first proved by de Jongh and Veltman [dJV99]. In our proof, the gain of assuringness becomes very clear.

1 Introduction

In this paper we present a generalization of the notion of a critical successor and show, in general terms and by an explicit example how this notion can be usefull in modal completeness proofs. In Section 2 we start with some conventions and notation. In Section 3 we introduce the notion of an assuring successor and address certain problems encountered in modal completeness proofs. In Section 4 project on the finite and in Section 5 we give a relatively simple proof of modal completeness and decidability of **IL**W.

2 Preliminaries

To get a compact presentation we assume the basic notions of interpretability logics. In particular we assume some familiarity with the notion of a critical successor and its use in modal completeness proofs [dJJ98][dJV90][GJ04].

In this paper we shall consider extensions of **IL** with the following principles.

$$
\begin{array}{lcl}
\mathsf{W} & := & A \rhd B \to A \rhd B \wedge \Box\neg A \\
\mathsf{M} & := & A \rhd B \to A \wedge \Box C \rhd B \wedge \Box C \\
\mathsf{P} & := & A \rhd B \to \Box(A \rhd B) \\
\mathsf{M}_0 & := & A \rhd B \to \Diamond A \wedge \Box C \rhd B \wedge \Box C \\
\mathsf{R} & := & A \rhd B \to \neg(A \rhd \neg C) \rhd B \wedge \Box C
\end{array}
$$

Uppercase Greek, like Γ and Δ, will denote maximal consistent sets (MCS's). It will be clear from the context with respect to what logic the consistency will refer. Uppercase Roman denotes modal interpretability formulas $A, B, C, \ldots$ or sets of such formulas $S, T, U, \ldots$. An exception to this rule is that we might write formulas form a set S as S_i, S_j etc. in particular if S is a set of formulas the $\bigvee S_i$ denotes a finite disjunction over some formulas in S. If we talk of logics we mean extensions of **IL**. As usual we use $\boxdot A$ as an abbriviation for $A \wedge \Box A$. If S is a set of formulas then we write $\Box S$ for $\{\Box A \mid A \in S\}$.

3 Extending criticality

In this section we will expose a generalization of critical successor and show how it can be used to solve, in a uniform way certain problematic aspects of modal completeness proofs.

Definition 3.1 (Assuring successor). Let S be a set of formulas. We define $\Gamma \prec_S \Delta$, and say that Δ is an *S-assuring successor* of Γ, if for any finite $S' \subseteq S$ we have $A \rhd \bigvee_{S_j \in S'} \neg S_j \in \Gamma \Rightarrow \neg A, \Box\neg A \in \Delta$ and for some $\Box C \in \Delta$ we have $\Box C \notin \Gamma$.

Lemma 3.2. *For the relation $\prec_S$ we have the following observations.*

1. $\Gamma \prec_\emptyset \Delta \Leftrightarrow \Gamma \prec \Delta$
2. *Δ is a B-critical successor of* $\Gamma \Leftrightarrow \Gamma \prec_{\{\neg B\}} \Delta$
3. $S \subseteq T$ & $\Gamma \prec_T \Delta \Rightarrow \Gamma \prec_S \Delta$
4. $\Gamma \prec_S \Delta \prec \Delta' \Rightarrow \Gamma \prec_S \Delta'$
5. $\Gamma \prec_S \Delta \Rightarrow S, \Box S \subseteq \Delta, \Diamond S \subseteq \Gamma$ *and for all A,* $\Diamond A \notin S$

Theorem 3.3. *Let Γ be a MCS and S a set of formulas. If for any choice of $S_i \in S$ we have that $\neg(B \rhd \bigvee \neg S_i) \in \Gamma$, then*[1] *there exists a MCS Δ such that $\Gamma \prec_S \Delta \ni B, \Box\neg B$.*

Proof. Suppose for a contradiction there is no such Δ. Then there is a formula[2] A such that for some $S_i \in S$, $(A \rhd \bigvee \neg S_i) \in \Gamma$ and $\Box\neg B, B, \Box\neg A, \neg A \vdash \bot$. Then $\vdash \Box\neg B \wedge B \rhd A \vee \Diamond A$ and we get $\vdash B \rhd A$. As $(A \rhd \bigvee \neg S_i) \in \Gamma$, also $(B \rhd \bigvee \neg S_i) \in \Gamma$. A contradiction. ⊣

Lemma 3.4. *Let Γ be a MCS such that $\neg(B \rhd C) \in \Gamma$. Then there is a MCS Δ such that $\Gamma \prec_{\{\neg C\}} \Delta$ and $B, \Box\neg B \in \Delta$.*

Proof. Taking $S = \{\neg C\}$ in Theorem 3.3. ⊣

[1] It is easy to see that we actually have iff.

[2] By compactness there are finitely many A_j with for some S_i^j, $(A_j \rhd \bigvee \neg S_i^j) \in \Gamma$ and $\Box\neg B, B\neg A_j, \Box\neg A_j \vdash \bot$. We can take A to be $\bigvee_j A_j$.

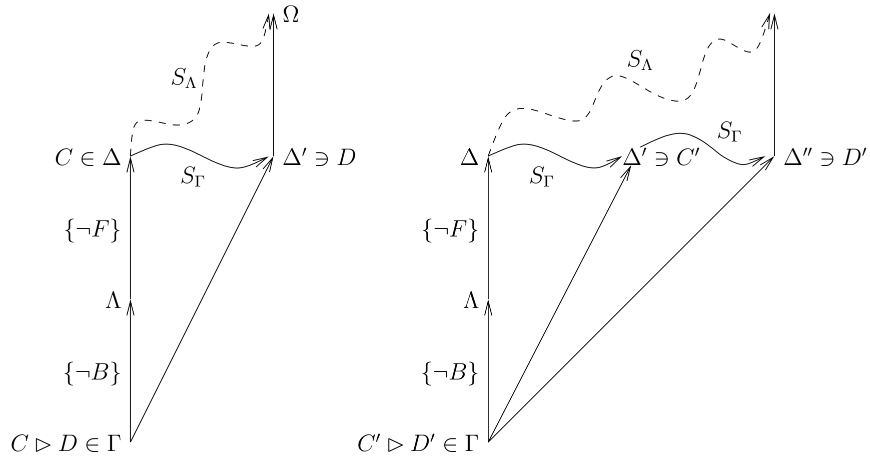

Figure 1: R frame condition

Lemma 3.5. *Let Γ and Δ be MCS's such that $A \rhd B \in \Gamma \prec_S \Delta \ni A$. Then there is a MCS Δ' such that $\Gamma \prec_S \Delta' \ni B, \Box\neg B$.*

Proof. First we see that for any choice of S_i, $\neg(B \rhd \bigvee \neg S_i) \in \Gamma$. Suppose not. Then for some S_i, $(B \rhd \bigvee \neg S_i) \in \Gamma$ because Γ is a MCS. But then $(A \rhd \bigvee \neg S_i) \in \Gamma$ and by $\Gamma \prec_S \Delta$ we have $\neg A \in \Delta$. A contradiction. So $\neg(B \rhd \bigvee \neg S_i) \in \Gamma$ for any choice of S_i and we can apply Theorem 3.3. ⊣

Lemmata 3.4, 3.5 are the obvious generalizations of the corresponding lemmata involving criticality instead of assuringness. To clarify the benefits of assuringness over criticality let us roughly identify the three main points when building a counter model $\langle W, R, S, V \rangle$ for some unprovable formula (in some extension of **IL**). We take W a multi set of MCS's and build the model in a step by step fashion.

1. For each $\Gamma \in W$ with $\neg(A \rhd B) \in \Gamma$ we should add some B-critical successor (equivalently $\{\neg B\}$-assuring successor) Δ to W for which $A \in \Delta$.
2. For each $\Gamma, \Delta \in W$ with $C \rhd D \in \Gamma R \Delta \ni C$ we should add a Δ' to W for which $\Gamma \prec \Delta' \ni D$. Moreover if Δ is a B-critical successor of Γ then then we should be able to choose Δ' a B-critical successor of Γ as well.
3. We should take care of the frame conditions.

When working in **IL**, Lemma 3.4 handles Item 1 and Lemma 3.5 handles Item 2. Making sure that the frame conditions are satisfied does not impose any problems [dJJ98]. With extensions of **IL** the situation regarding the frame conditions becomes more complicated [dJV90][GJ04]. Let us clarify this by looking at **IL**R. The additional frame condition is as follows [GJ04][3].

$$wRxRyS_wy'Rz \Rightarrow yS_xz$$

This is depicted in the leftmost picture in Figure 1. Let us use the notation as in Item 2: Δ' was added to the model since $C \rhd D \in \Gamma R \Delta \ni C$. Since Δ lies F-critical (equivalently $\{\neg F\}$-assuring) above Λ, we should not only make sure that Δ' lies B critical above Γ, but also that for any successor Ω of Δ' lies F critical above Λ.

One way to guarantee this is to actually require that $\Box\neg H \in \Delta'$ whenever $H \rhd F \in \Lambda$. As one easily checks, it is quite easy to prove such a Lemma in **IL**R but we have oversimplified[4] the situation. Consider the rightmost picture in Figure 1. That is, after having added Δ' to the model we are required to add some Δ'' with $D' \in \Delta'$ to the model since $C' \rhd D' \in \Gamma$ and $C' \in \Delta'$. By the transitivity of S_Γ we require that $\Box\neg H \in \Delta''$ whenever $H \rhd F \in \Lambda$. In this situation it is not so clear what to do.

Although for **IL**$\mathsf{M_0}$ [GJ04] and **IL**W [dJV99] there where add hoc solutions to similar problems, criticality seemed too weak a notion for a more uniform solution. As the lemmata below will show, assuringness does give us a uniform method for handling these kind of situations.

In what follows put, for any set of formulas T,

$$\Delta^{\Box}_T = \{\Box\neg A \mid T' \subseteq T \text{ finite }, A \rhd \bigvee_{T_i \in T'} \neg T_i \in \Delta\},$$
$$\Delta^{\boxdot}_T = \{\Box\neg A, \neg A \mid T' \subseteq T \text{ finite }, A \rhd \bigvee_{T_i \in T'} \neg T_i \in \Delta\}.$$

Lemma 3.6. *For any logic (i.e. extension of* **IL***) we have* $\Gamma \prec_S \Delta \Rightarrow \Gamma \prec_{S \cup \Gamma^{\boxdot}_S} \Delta$.

Proof. Suppose $\Gamma \prec_S \Delta$ and $C \rhd \bigvee \neg S_i \vee \bigvee A_j \vee \Diamond A_j \in \Gamma$. Then $C \rhd \bigvee \neg S_i \vee \bigvee A_j \in \Gamma$ and thus $C \rhd \bigvee \neg S_i \vee \bigvee \neg S^j_k \in \Gamma$ which implies $\neg C, \Box\neg C \in \Delta$. ⊣

Lemma 3.7. *For logics containing* M *we have* $\Gamma \prec_S \Delta \Rightarrow \Gamma \prec_{S \cup \Delta^{\Box}_{\emptyset}} \Delta$.

Proof. Note that $\Delta^{\Box}_{\emptyset} = \{\Box C \mid \Box C \in \Delta\}$. We consider A such that for some $S_i \in S$ and $\Box C_j \in \Delta^{\Box}_{\emptyset}$, $(A \rhd \bigvee \neg S_i \vee \bigvee \neg \Box C_j) \in \Gamma$. By M, $(A \wedge \bigwedge \Box C_j \rhd \bigvee \neg S_i) \in \Gamma$, whence $\boxdot\neg(A \wedge \bigwedge \Box C_j) \in \Delta$. As $\bigwedge \Box C_j \in \Delta$, we conclude $\neg A, \Box\neg A \in \Delta$. ⊣

Lemma 3.8. *For logics containing* P *we have* $\Gamma \prec_S \Lambda \prec_T \Delta \Rightarrow \Gamma \prec_{S \cup \Lambda^{\boxdot}_T} \Delta$.

[3] In [GJ04] the modal principle $A \rhd B \to \neg(A \rhd \neg C) \wedge (D \rhd C) \rhd B \wedge \Box C$ was called R. This principle and the one called R in this paper are eaily seen to be equivalent over **IL**.

[4] The reader should note that we do not give a completeness proof for **IL**R here. We only indicate a few problems one will encounter and indicate the usefulness of assuringness by overcoming these. In general assuring-ness does not yet give the answere to all problems encountered in modal completeness proofs. However, in the special case of **IL**RW assuring-ness can be put to use to give a completeness proof.

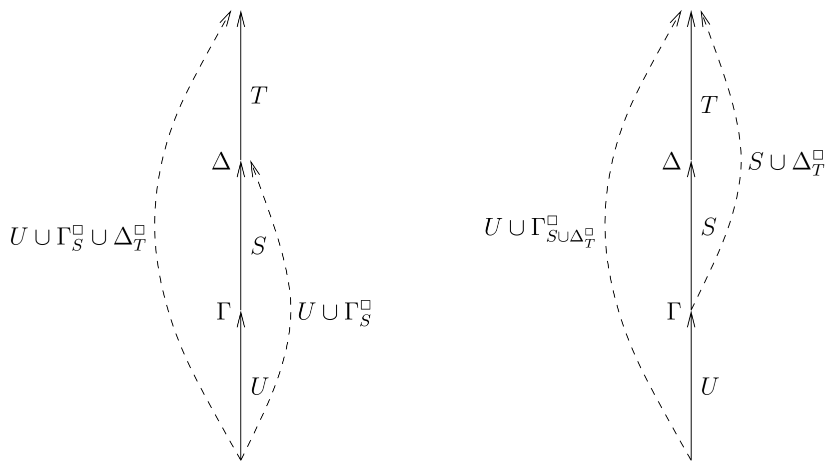

Figure 2: Two ways for computing the transitive closure in **IL**R.

Proof. Suppose $C \rhd \bigvee \neg S_i \vee \bigvee A_j \vee \Diamond A_j \in \Gamma$, where $\Box\neg A_j, \neg A_j \in \Delta^{\boxdot}_T$. Then $C \rhd \bigvee \neg S_i \vee \bigvee A_j \in \Gamma$ and thus by P we obtain $C \rhd \bigvee \neg S_i \vee \bigvee A_j \in \Lambda$. Since $\Gamma {\prec_S} \Lambda$ we have $\Box \bigwedge S_i \in \Lambda$ so we obtain $C \rhd \bigvee A_j \in \Lambda$. But for each A_j we have $A_j \rhd \bigvee \neg T_{jk} \in \Lambda$ and thus $C \rhd \bigvee T_{jk} \in \Lambda$. Since $\Lambda \prec_T \Delta$ we conclude $\neg C, \Box\neg C \in \Delta$. $\dashv$

Lemma 3.9. *For logics containing* M_0 *we have* $\Gamma {\prec_S} \Delta \prec \Delta' \Rightarrow \Gamma {\prec_{S \cup \Delta^{\Box}_{\emptyset}}} \Delta'$.

Proof. Suppose $C \rhd \bigvee S_i \vee \bigvee \Diamond A_j \in \Gamma$, where $\Box\neg A_j \in \Delta^{\Box}_{\emptyset}$. By M_0 we obtain $\Diamond C \wedge \bigwedge \Box\neg A_j \rhd \bigvee S_i \in \Gamma$. So, since $\Gamma {\prec_S} \Delta$ and $\bigwedge \Box\neg A_j \in \Delta$ we obtain $\Box\neg C \in \Delta$ and thus $\Box\neg C, \neg C \in \Delta'$. $\dashv$

Lemma 3.10. *For logics containing* R *we have* $\Gamma {\prec_S} \Delta {\prec_T} \Delta' \Rightarrow \Gamma {\prec_{S \cup \Delta^{\Box}_T}} \Delta'$.

Proof. We consider A such that for some $S_i \in S$ and some $\Box\neg A_j \in \Delta^{\Box}_T$, we have $(A \rhd \bigvee \neg S_i \vee \bigvee \Diamond A_j) \in \Gamma$. By R we obtain $(\neg(A \rhd \bigvee A_j) \rhd \bigvee \neg S_i) \in \Gamma$, thus by $\Gamma {\prec_S} \Delta$ we get $(A \rhd \bigvee A_j) \in \Delta$. As $(A_j \rhd \bigvee \neg T_{kj}) \in \Delta$, also $(A \rhd \bigvee \neg T_{kj}) \in \Delta$. By $\Delta {\prec_T} \Delta'$ we conclude $\boxdot\neg A \in \Delta'$. $\dashv$

What lemmata 3.8, 3.9 and 3.10 actually tell us is how to label R relations when we take R transitive while working in the lemma's respective logic. However, there is an easily identifiable problem here. Suppose we are working in **IL**R. Consider the two pictures in Figure 2. If we compute the label between the lower world and the upper world it does make a difference whether we first compute the label between the lower world and Δ (left picture) or the label between Γ and the upper world (right picture). We will show in Lemma 3.11 below that in the situation as given in Figure 2 we have

$$U \cup \Gamma^{\Box}_{S \cup \Delta^{\Box}_T} \subseteq U \cup \Gamma^{\Box}_S \cup \Delta^{\Box}_T.$$

And we should thus opt for the strategy as depicted in the leftmost picture when computing the transitive closure of R.

Lemma 3.11. *For logics containing*[5] R *we have* $\Gamma \prec_S \Delta \Rightarrow \Gamma^{\Box}_{S \cup \Delta^{\Box}_{T}} \subseteq \Delta^{\Box}_{T}$.

Proof. Consider $\Box\neg A \in \Gamma^{\Box}_{S\cup\Delta^{\Box}_{T}}$, that is, for some $S_i \in S$ and $\Box\neg B_j \in \Delta^{\Box}_{T}$, $A \rhd \bigvee \neg S_i \vee \bigvee \neg\Box\neg B_j \in \Gamma$. By R, $\neg(A \rhd \bigvee B_j) \rhd \bigvee \neg S_i \in \Gamma$, whence by $\Gamma \prec_S \Delta$, we get $A \rhd \bigvee B_j \in \Delta$. But for each B_j there is $T_{jk} \in T$ with $B_j \rhd \bigvee \neg T_{jk} \in \Delta$, whence $A \rhd \bigvee \neg T_{jk} \in \Delta$ and $\Box\neg A \in \Delta^{\Box}_{T}$. ⊣

Lemmata as Lemma 3.7, 3.9 and 3.10 are what we call *labeling lemma*. We propose the following slogan.

> **Slogan:** Every complete logic with a first order frame condition has its own labeling lemma.

Let us state two lemmata for **IL**W, a logic without a first order frame property. As predicted by our slogan, these do not fit in very nicely with the previous ones.

Lemma 3.12. *Suppose* $\neg(A \rhd B) \in \Gamma$. *There exists some* Δ *with* $\Gamma \prec_{\{\Box\neg A, \neg B\}} \Delta$ *and* $A \in \Delta$.

Proof. Suppose for a contradiction that there is no such Δ. Then there is a formula E with $(E \rhd \Diamond A \vee B) \in \Gamma$ such that $A, \neg E, \Box\neg E \vdash \bot$ and so $\vdash A \rhd E$. Then $(A \rhd \Diamond A \vee B) \in \Gamma$ and by the principle W we have $A \rhd B \in \Gamma$. The contradiction. ⊣

Lemma 3.13. *For logics containg* W *we have that if* $B \rhd C \in \Gamma \prec_S \Delta \ni B$ *then there exists* Δ *with* $\Gamma \prec_{S \cup \{\Box\neg B\}} \Delta \ni C, \Box\neg C$.

Proof. Suppose for a contradiction that no such Δ exists. Then for some formula A with $(A \rhd \bigvee \neg S_i \vee \Diamond B) \in \Gamma$, we get $C, \Box\neg C, \neg A, \Box\neg A \vdash \bot$, whence $\vdash C \rhd A$. Thus $B \rhd C \rhd A \rhd \bigvee \neg S_i \vee \Diamond B \in \Gamma$. By W, $B \rhd \bigvee \neg S_i \in \Gamma$ which contradicts $\Gamma \prec_S \Delta \ni B$. ⊣

4 Going finite

Proving the decidability of an interpretabilty logic is in all known cases done by showing that the logic has the finite model property. The finite model property is easier to achieve if the building blocks of the model are finite sets instead of infinite maximal consistent sets.

A turn that is usually made to obtain finite building blocks, is to work with truncated parts of maximal consistent sets. This part should be large enough to allow for the basic reasoning. This gives rise to the notion of so-called adequate sets where different logics yield different notions of adequacy. In order to obtain the finite model property along with modal completeness of **IL**W, in the next section we will use the following notion of adequacy.

[5]For the other logics we get similar lemmata.

Definition 4.1 (Adequate set). We say that a set of formulas Φ is *adequate* iff

1. $\bot \rhd \bot \in \Phi$
2. Φ is closed under single negation and subformulas
3. If both A is an antecedent or consequent of some $\rhd$ formula in Φ and so is B then $A \rhd B \in \Phi$

It is clear that any formula is contained in some finite and minimal adequate set. For a formula F we will denote this set by $\Phi(F)$. Since our maximal consistent sets are more restricted we should also modify the notion of an assuring successor a bit

Definition 4.2 ($\langle S, \Phi\rangle$-assuring successor). Let Φ be a finite adequate set, $S \subseteq \Phi$ and $\Gamma, \Delta \subseteq \Phi$ be maximal consistent sets. We say that Δ is an $\langle S, \Phi\rangle$-assuring successor of Γ ($\Gamma \prec_S^\Phi \Delta$) iff for each $\Box\neg A \in \Phi$ we have

$$\Gamma \vdash A \rhd \bigvee_{S_i \in S} \neg S_i \Rightarrow \neg A, \Box\neg A \in \Delta.$$

Moreover for some $\Box C \in \Delta$ we have $\Box C \notin \Gamma$.

Note that by the requirement $\Box\neg A \in \Phi$ the usual reading of $\prec$ in extensions of **GL** coincides with $\prec_\emptyset^\Phi$. So we will write $\prec$ for $\prec_\emptyset^\Phi$. The following two lemmata are proved exactly as their infinite counterparts.

Lemma 4.3. *Let $\Gamma \subseteq \Phi$ be maximal consistent. If $\neg(A \rhd B) \in \Gamma$ then there exists some maximal consistent set $\Delta \subseteq \Phi$ such that $A \in \Delta$ and $\Gamma \prec_{\{\neg B, \Box\neg A\}}^\Phi \Delta$.*

Lemma 4.4. *Let $\Gamma, \Delta \subseteq \Phi$ be maximal consistent and $S \subseteq \Phi$. If $A \rhd B \in \Gamma$, $\Gamma \prec_S^\Phi \Delta$ and $A \in \Delta$ then there exists some maximal consistent $\Delta' \subseteq \Phi$ with $B \in \Delta'$ and $\Gamma \prec_{S \cup \{\Box\neg A\}}^\Phi \Delta'$.*

5 The logic ILW

As a demonstration of the use of assuringness we will give in this section a relatively simple proof of the known fact that **IL**W is a complete logic.

In what follows we let Φ be some fixed finite adequate set and reason with **IL**W (e.g. $\vdash$ is **IL**W-provable, and consistent is **IL**W-consistent). The rest of this section is devoted to the proof of the following theorem.

Theorem 5.1 (Completeness of ILW **[dJV99]).** **IL**W *is complete with regspect to finite Veltman frames $\langle W, R, S\rangle$ in which, for each $w \in W$, $(S_w; R)$ is conversely well-founded (c.w.f.).*

Suppose $\nvdash G$. Let $\Phi = \Phi(\neg G)$ and let $\Gamma \subseteq \Phi$ be a maximal consistent set that contains $\neg G$. We will construct a Veltman model $\langle W, R, S, V\rangle$ in which for each $w \in W$ we have that $(S_w; R)$ is conversely well-founded. Each $w \in W$ will be a tuple the second component, denoted by $(w)_1$, of which will be a maximal consistent subset of Φ. For some $w \in W$ we will have $(w)_1 = \Gamma$ and we will finish the proof by proving a truth lemma: $w \Vdash A$ iff $A \in (w)_1$.

Let the *height* of a maximal consistent $\Delta \subseteq \Phi$ be defined as the number of $\Box$-formulas in Δ minus the number of $\Box$-formulas in Γ. For sequences σ_0 and σ_1 we write $\sigma_0 \subseteq \sigma_1$ iff σ_0 is an initial, but not necessarily proper subsequence of σ_1. For two sequences σ_0 and σ_1, $\sigma_0 * \sigma_1$ denotes the concatenation of the two sequences. If S is a set of formulas then $\langle S \rangle$ is the sequence of lenght one and only element S. Let us now define $\langle W, R, S, V \rangle$.

1. W is the set of tuples $\langle \sigma, \Delta \rangle$ where $\Delta \subseteq \Phi$ is maximal consistent such that either $\Gamma = \Delta$ or $\Gamma \prec \Delta$ and σ is a finite sequence of subsets of Φ the length of which does not exceed the height of Δ. For $w = \langle \sigma, \Delta \rangle$ we write $(w)_0$ for σ and $(w)_1$ for Δ.

2. wRv iff for some S we have $(v)_0 \supseteq (w)_0 * \langle S \rangle$ and $(w)_1 \prec_S^\Phi (v)_1$.

3. xS_wy iff wRx, y and, xRy or $x = y$ or both 3a and 3b hold.

   (a) If $(x)_0 = (w)_0 * \langle S \rangle * \tau_x$, $(y)_0 = (w)_0 * \langle T \rangle * \tau_y$ then $S \subseteq T$

   (b) For some $C \rhd D \in (w)_1$ we have $\Box\neg C \in T$ and, $C \in (x)_1$ or $\Diamond C \in (x)_1$

4. $V(p) = \{w \in W \mid p \in (w)_1\}$.

Lemma 5.2. *R is transitive and conversely well-founded.*

Proof. Transitivity follows from the fact that $(x)_1 \prec_S^\Phi (y)_1 \prec (z)_1$ implies $(x)_1 \prec_S^\Phi (z)_1$. Conversely well-foundedness now follows from the fact that our model is finite and R is irreflexive. ⊣

Lemma 5.3. *$wRxRy$ implies xS_wy, wRx implies xS_wx, S_w is transitive*

Proof. The first two assertions hold by definition. So suppose xS_wyS_wz. Let us fix $(x)_0 \supseteq (w)_0 * \langle S \rangle$, $(y)_0 \supseteq (w)_0 * \langle T \rangle$ and $(z)_0 \supseteq (w)_0 * \langle U \rangle$. We distinguish two cases.

Case 1: xRy or $x = y$. If $x = y$ then we are done so we assume xRy. If yRz or $y = z$ then we are also easily done. So, we assume that for some $C \rhd D \in (w)_1$ we have $\Box\neg C \in U$ and, $C \in (y)_1$ or $\Diamond C \in (y)_1$. Since $(x)_1 \prec (y)_1$ we have that $\Diamond C \in (x)_1$ and thus we conclude xS_wz.

Case 2: $\neg xRy$ and $x \neq y$. In this case there exists some $C \rhd D \in (w)_1$ with $\Box\neg C \in T$ and $C \in (x)_1$ or $\Diamond C \in (x)_1$. Whatever the reason for yS_wz is, we always have $T \subseteq U$ and thus $\Box\neg C \in U$. So we conclude xS_wz.

⊣

Lemma 5.4. *$(S_w; R)$ is conversely well-founded.*

Proof. Suppose we have an infinite sequence

$$x_0 S_w y_0 R x_1 S_w y_1 R \cdots .$$

For each $i \geq 0$, fix X_i and Y_i such that $(x_i)_0 \supseteq (w)_0 * \langle X_i \rangle$ and $(y_i)_0 \supseteq (w)_0 * \langle Y_i \rangle$. We may assume that, for each i, $x_i \neq y_i$ and $\neg x_i R y_i$. Thus, let $C_i \rhd D_i$ be the formula as given by condition 3b. We thus have $C_i \rhd D_i \in (w)_1$, $\Box\neg C_i \in Y_i$ and, $C_i \in (x_i)_1$ or $\Diamond C_i \in (x_i)_1$. For any $j > i$, this implies $\Box\neg C_i \in X_j$ which gives $\Box\neg C_i \in (x_j)_1$ and thus $\neg C_i, \Box\neg C_i \in (y_j)_1$. The latter gives $C_i \neq C_j$, which is a contradiction since Φ is finite. ⊣

Lemma 5.5 (Truth lemma). *For all $F \in \Phi$ and $w \in W$ we have $F \in (w)_1$ iff $w \Vdash F$.*

Proof. We proceed by induction on F.

The cases of the propositional variables and the connectives are easily provable using properties of MCS's and the $\Vdash$ relation. So suppose $F = A \rhd B$.

($\Rightarrow$) Suppose we have $A \rhd B \in (w)_1$. Then for all v such that wRv and $v \Vdash A$ we have to find a u such that $vS_w u \Vdash B$ which, by the induction hypothesis, is equivalent to $B \in (u)_1$. Consider such a v. We have for some S that $(v)_0 = (w)_0 * \langle S \rangle * \tau$ and $(w)_1 \prec_S^\Phi (v)_1$. By Lemma 4.4 there is a MCS Δ such that $(w)_1 \prec^\Phi_{S\cup\{\Box\neg A\}} \Delta$. We take $u = \langle (w)_0 * \langle S \cup \{\Box\neg A\}\rangle, \Delta\rangle$. Now 3b holds whence $vS_w u$.

($\Leftarrow$) Suppose that $A \rhd B \notin (w)_1$. Then $\neg(A \rhd B) \in (w)_1$ whence by Lemma 4.3 there is a MCS Δ such that $(w)_1 \prec^\Phi_{\{\Box\neg A, \neg B\}} \Delta \ni A$. Consider $v' = \langle (w)_0 * \langle \{\Box\neg A, \neg B\}\rangle * \tau, \Delta\rangle$. Clearly, there is no u' such that $v' S_w u' \Vdash B$. ⊣

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