The case of the hidden hand

Hans van Ditmarsch, University of Otago, New Zealand
hans@cs.otago.ac.nz

From a pack of seven known cards two players each draw three cards and a third player gets the remaining card. How can the players with three cards openly inform each other about their cards, without the third player learning from any of their cards who holds it?

This 'Russian Cards' problem originated at the Moscow Math Olympiad 2000. An analysis in dynamic epistemic logic and various solutions are presented in [vD03]. But there remained some open questions related to the dynamics. In this contribution we answer one of those questions. They are not trivial, because the interpretation of an announcement that is made towards a solution of the problem, also depends on the commonly known intentions of rational agents executing a protocol resulting in such an announcement. These intentions, standardly delegated to the *pragmatics* of communication, can be drawn into the *semantics* of utterances. In plain words: what you mean, is more than what you say.

Call the players Anne, Bill and Cath, and the cards $0, ..., 6$, and suppose Anne holds $\{0, 1, 2\}$, Bill $\{3, 4, 5\}$, and Cath card 6. For the hand of cards $\{0, 1, 2\}$, write 012 instead, for the card deal, write 012.345.6, etc. All announcements must be public and truthful. There are not many things Anne can safely say. Obviously, she cannot say "I have 0 or 6," because then Cath learns that Anne has 0. But Anne can also not say "I have 0 or 3," because Anne does not know if Cath has 3 or another card, and *if* Cath had card 3, she would have learnt that Anne has card 0. But Anne can also not say "I have 0 or 1." Even though Anne holds both 0 and 1, so that she does not appear to risk that Cath eliminates either card and thus gains knowledge about single card ownership (weaker knowledge, about alternatives, is allowed), Cath knows that Anne will not say anything from which Cath may learn her cards. And thus Cath can conclude that Anne will only say "I have 0 or 1" if she actually holds both 0 and 1. And in that way Cath learns two cards at once! The apparent contradiction between Cath not knowing and Cath knowing is not really there, because these observations are about different information states: it is merely the case that announcements may induce further updates that contain yet other information. There are various solutions that consist of first Anne and then Bill making an announcement, but – just to challenge the reader – all of the following are no solutions and run into trouble of the aforementioned kind (for details, see [vD03]): *Anne says that either she or Bill holds 012, after which Bill says that either he or Anne holds 345*, and also *Anne says that she does not hold card 6, after which Bill says that he does not hold card 6 either*, and also *Anne says that she either holds 012 or not any of those cards, after which Bill says that Cath holds card 6.* In all those cases, it turns out that, already after Anne's announcement, it is (at least) not common knowledge that Cath is ignorant of any of Anne's or Bill's cards, and that this is informative to Cath. Indeed, the solution requirement should be that Cath's ignorance remains common knowledge after any announcement. Such announcements are called *safe*. Further, one can prove that all informative announcements are equivalent to one of the form "my hand of cards is one of the following alternatives," so that all solutions consist of alternating statements of the players in that form. A combinatorial equivalent for a safe announcement consisting of hands, is (restricted to the set of cards that are not publicly known to be held by Cath): for each card, in the set of hands not containing that card, all other cards occur in at least one hand and are absent in at least one hand. A solution to the Russian Cards problem is a sequence of safe announcements after which it is commonly known that Anne knows Bill's hand and Bill knows Anne's hand. The following is a solution:

> *Anne says "My hand of cards is one of 012, 034, 056, 135, 246,"*
> *after which Bill says "Cath has card 6."*

Note that Bill's announcement is equivalent to "My hand of cards is one of 345, 125, 024." After Anne's announcement, Bill knows Anne's hand because one of his cards 3, 4, and 5 occurs in all hands except 012. After Bill's announcement, Anne knows Bill's hand as well, as 3, 4, and 5 are the remaining cards not held by Cath. After both Anne's and Bill's announcement, it is common knowledge that Cath does not know any of their cards. This can be proven by checking the combinatorial requirements. For example, after Anne's announcement, if Cath holds 0, the remaining hands are 135 and 246. Each of the cards $1, 2, ..., 6$ both occurs in at least one of 135 and 246 and is absent in at least one of those: 1 occurs in 135 and is absent in 246, 2 occurs in 246 and is absent in 135, etc. If Cath holds card 1, etc. After Bill's announcement this check is even easier. So both announcements are safe. Also, after both announcements it is common knowledge that Anne knows Bill's hand, and vice versa.

If we *remove a single hand* from Anne's announcement, it can easily be seen that Cath will learn one or more of Anne's cards. For example, let us remove 246. Cath can now reason as follows: "Suppose Anne does not have 0. Then Anne can imagine that I have 0, in which case I could have eliminated all but 135 and learnt her cards. So if she does not have 0, she would never have said that. But she just did. So she must have 0. So I learnt one of her cards after all!" Similarly, Cath can now conclude that Anne holds 1, and so learns Anne's entire hand. If one removes another hand instead, a similar argument follows.

Now consider what happens if we *add a single hand* to Anne's announcement, a 'hidden hand' so to speak, as without it, her announcement already served its purpose; it therefore appears to be unrelated to the protocol underlying the previous announcement. For example, we add 245:

> *Anne says "My hand of cards is one of 012, 034, 056, 135,* **245**, *246," after which Bill says "Cath has card 6."*

As Anne's announcement is now slightly weaker, it is tempting to conclude that it is therefore less informative than her previous announcement. But is this really so?

First, let us assume that it is common knowledge to all players that after both announcements 'the problem will be solved', or, in other words, that the underlying protocol is of length two. On this assumption Cath actually learns some of Anne's cards: If Anne holds 245 or 246, then Anne can imagine (does not know not) that Bill has not learnt her hand, namely if Bill holds 013. Therefore, if the solution is known in advance to be of length two, Anne does not hold 245 or 246, but one of 012 034 056 135. Cath knows all of that too. But that is precisely the four hand announcement just discussed. That was proven unsafe: Cath learnt Anne's entire hand of cards! So she will now, again. We see that instead of being *less* informative, Anne's six hand announcement is actually *more* informative than her five hand announcement. This is because Cath can assume that Anne's six hand announcement must have been informative enough for Bill to learn her cards.

Next, suppose that we do not assume that the underlying protocol is of length two. Even though Anne knows that Bill knows her hand of cards, Cath can imagine (does not know not) that Anne does not know that: Cath, who holds 6, can imagine that Anne holds 245 and Bill 013, in which case Bill would not have learnt Anne's hand, so that a fortiori Cath can imagine that Anne can imagine that Bill has not learnt Anne's hand. Other choices of the sixth hand give slightly different results, but it always follows that it is not commonly known that Bill knows Anne's hand. On the other hand, we can compute in a way similar to that for the five hand protocol, that both after Anne's and after Bill's announcement it now remains common knowledge that Cath does not know any of Anne's or Bill's cards, and that after Bill's it is common knowledge that Anne knows Bill's hand and Bill knows Anne's. So, on the assumption that it is not commonly known that the protocol is of length two, we have found a solution of the Russian Cards problem of length two. This solution is different from the previous solution, because the intermediate information states are different: after Anne's announcement in the first sequence it is common knowledge that Anne knows Bill's cards, but after Anne's announcement in the second sequence this is not common knowledge. So we have found a new solution to the Russian Cards Problem![1]

Or haven't we?

The 'hidden hand' 245 – hidden because it appears not to be actually used in the

[1] The six hand example and this complication were suggested by Sieuwert van Otterloo.

protocol – only makes sense, if, when not 012.345.6 but instead 245.013.6 were the actual deal, there is a continuation of the communication between Anne and Bill, starting with Anne's six hand announcement, that also results in the solution requirements. Because if not, and because all three players are rational, then that hand 245 can be ruled out after all from public consideration, and both sequences would then be 'essentially' the same, i.e., describing identical information state transitions. This requires a systematic investigation of *all* possible continuations of that dialogue, which is exactly what we will undertake in this contribution. It turns out we only need to investigate protocols up to length four. But before we continue the exposition, we introduce the logic of public announcements in which this discussion finds a convenient and much more intelligable formal setting, so that we can do without the precise but sometimes confusing descriptions in natural language that we have used so far.

Given a set of *agents* N and a set of (propositional) *atoms* P, our basic structure is the *epistemic model* $M = \langle S, \sim, V \rangle$, where S is a *domain* of (factual) *states*, $\sim : N \rightarrow \mathcal{P}(S \times S)$ defines a set of *accessibility* relations $\sim_n$ that are equivalence relations, and $V : P \rightarrow \mathcal{P}(S)$ defines a set of *valuations* $V_p \subseteq S$. A pointed structure (M, s) is called an *epistemic state*. The logical language is inductively defined as

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_n\varphi \mid C_N\varphi \mid [\varphi]\psi$$

For $K_n\varphi$ read 'agent n knows φ', for $C_N\varphi$ read '(the public) N commonly know φ', and for $[\varphi]\psi$ read 'after (every) announcement of φ, it holds that ψ'. For the dual $\neg K_n \neg\varphi$ of 'knowing that', read 'agent n can imagine that φ', and we also write $\hat{K}_n\varphi$ for that. The semantics of this *multiagent logic of public announcements* is

$$\begin{array}{lll} M,s \models p & : \text{iff} & s \in V_p \\ M,s \models \neg\varphi & : \text{iff} & M,s \not\models \varphi \\ M,s \models \varphi \wedge \psi & : \text{iff} & M,s \models \varphi \text{ and } M,s \models \psi \\ M,s \models K_n\varphi & : \text{iff} & \text{for all } t \sim_n s : M,t \models \varphi \\ M,s \models C_N\varphi & : \text{iff} & \text{for all } t \sim_N s : M,t \models \varphi \\ M,s \models [\varphi]\psi & : \text{iff} & M,s \models \varphi \text{ implies } M|\varphi, s \models \psi \end{array}$$

where $\sim_N$ is the reflexive and transitive closure of the union of all $\sim_n$, i.e., $\sim_N := (\bigcup_{n \in N} \sim_n)^*$, and $M|\varphi$ is the restriction of M to the states where φ is true, i.e., $M|\varphi := \langle S', \sim', V' \rangle$ such that

$$\begin{array}{lll} S' & := & \{v \in S \mid M, v \models \psi\} \\ \sim'_n & := & \sim_n \cap (S' \times S') \\ V'_p & := & V_p \cap S' \end{array}$$

From the various principles that hold for this logic, we merely mention two validities that we will refer to in the continuation: $[C_N\varphi]C_N\varphi$ says that the announcement of something that is already publicly known is not informative, and $[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$ says that if you first announce φ and after that ψ, you might as well have announced all at once $\varphi \wedge [\varphi]\psi$. For a proof system, and a reference to a completeness proof, see [BM04], of which this logic is a special case. There is no proper reference – yet – just treating this logic.

For the Russian Cards example, there are three agents and 21 atoms (seven cards times three agents). Atom q_n describes the fact that agent n holds card q, and $ijk_n := i_n \wedge j_n \wedge k_n$ describes that player n's hand is $\{i, j, k\}$, so that Anne holding card 0 is described by 0_a, and, that Anne's hand is 012, is described by 012_a, etc. The structures on which we interpret such descriptions consist of a domain containing all deals of cards $Q = \{0, 1, 2, 3, 4, 5, 6\}$ over players $N = \{a, b, c\}$ (for Anne, Bill, and Cath, respectively). The equivalences on this domain are induced by players being able to see their own cards, and how many cards other players have. Therefore we can restrict ourselves to the (connected) model consisting of the $\binom{7}{3} \cdot \binom{4}{3} \cdot \binom{1}{1} = 140$ card deals where Anne and Bill each hold three and Cath one card. We call this model Rus (for 'Russian'). E.g., the a-equivalence class of deal 012.345.6 is $\{012.345.6, 012.346.5, 012.356.4, 012.456.3\}$, whereas the b-equivalence class of that deal is $\{012.345.6, 016.345.2, 026.345.1, 126.345.0\}$, and its c-equivalence class contains $\binom{6}{3} = 20$ card deals. The epistemic requirements for a problem solution are:

$$\begin{array}{lcl} \mathsf{aknowsbs} & := & \bigwedge_{i \neq j \neq k \in Q} (ijk_b \rightarrow K_a ijk_b) \\ \mathsf{bknowsas} & := & \bigwedge_{i \neq j \neq k \in Q} (ijk_a \rightarrow K_b ijk_a) \\ \mathsf{cignorant} & := & \bigwedge_{q \in Q} \bigwedge_{n=a,b} \neg K_c q_n \end{array}$$

When an agent n is saying φ, this is interpreted as the announcement of $K_n\varphi \wedge [K_n\varphi]C_N\mathsf{cignorant}$. Using the validities above, we see that $[K_n\varphi \wedge [K_n\varphi]C_N\mathsf{cignorant}]\psi$ is equivalent to $[K_n\varphi][C_N\mathsf{cignorant}]\psi$. Using the validity of $[C_N\mathsf{cignorant}]C_N\mathsf{cignorant}$[2], we can characterize a 'safe announcement' as one that is true and after which $\mathsf{cignorant}$ is common knowledge. A solution is a sequence of safe announcements after which $C_N(\mathsf{aknowsbs} \wedge \mathsf{bknowsas})$ is true.

We now can formalize the difference between the five hand and the six hand solution that we investigate. Define

$$\begin{array}{lcl} \mathsf{anne5} & := & K_a(012_a \vee 034_a \vee 056_a \vee 135_a \vee 246_a) \\ \mathsf{anne6} & := & K_a(012_a \vee 034_a \vee 056_a \vee 135_a \vee 245_a \vee 246_a) \\ \mathsf{bill} & := & K_b 6_c \end{array}$$

These announcements are all safe. The two solution sequences can be abbreviated as $\mathsf{anne5}; \mathsf{bill}$ and $\mathsf{anne6}; \mathsf{bill}$. Their difference appears from the models $Rus|\mathsf{anne5}$ and $Rus|\mathsf{anne6}$. The model $Rus|\mathsf{anne5}$ can be pictured as

012.345.6	012.346.5	012.356.4	012.456.3				
034.125.6	034.126.5			034.156.2	034.256.1		
		056.123.4	056.124.3	056.134.2	056.234.1		
135.024.6		135.026.4		135.046.2		135.246.0	
	246.013.5		246.015.3		246.035.1	246.135.0	

[2]Schema $[\varphi]\varphi$ is not valid in for all φ. The last of the introductory examples is *not* a solution because $Rus|K_a(012_a \vee \neg(0_a \vee 1_a \vee 2_a)), 012.345.6 \models [K_a\mathsf{cignorant}]\neg K_a\mathsf{cignorant}$.

The rows are a-equivalence classes, all b-equivalence classes are singleton, and the columns are c-equivalence classes. For example, $Rus|\mathsf{anne5}, 012.345.6 \models C_{abc}\mathsf{bknowsas}$, because all b-equivalence classes are singleton. The model $Rus|\mathsf{anne6}$ can be pictured as

```
012.345.6 012.346.5 012.356.4 012.456.3
034.125.6 034.126.5                     034.156.2 034.256.1
                    056.123.4 056.124.3 056.134.2 056.234.1
135.024.6           135.026.4           135.046.2           135.246.0
245.013.6*                    245.016.3           245.036.1 245.136.0
          246.013.5*          246.015.3           246.035.1 246.135.0
```

where all b-equivalence classes are singleton except $\{245.013.6, 246.013.5\}$. Those deals therefore appear with a *. We now have that $Rus|\mathsf{anne5}, 012.345.6 \models \hat{K}_c\neg K_a\mathsf{bknowsas}$, because $012.345.6 \sim_c 245.013.6$ and $Rus|\mathsf{anne5}, 245.013.6 \models \neg K_a\mathsf{bknowsas}$, because $245.013.6 \sim_a 245.013.6$ and $Rus|\mathsf{anne5}, 245.013.6 \models \neg\mathsf{bknowsas}$. The last is true, because 245_a holds but not $K_b 245_a$ (because $245.013.6 \sim_b 246.013.5$ and $Rus|\mathsf{anne5}, 246.013.5 \not\models 245_a$), so that we have $Rus|\mathsf{anne5}, 245.013.6 \not\models 245_a \rightarrow K_b 245_a$. Both after $(\mathsf{anne5}; \mathsf{bill})$ and after $(\mathsf{anne6}; \mathsf{bill})$ the model is

```
012.345.6
034.125.6
135.024.6
```

in which all solution requirements are common knowledge. This will be enough backbone to strengthen our exposition after having explained the further developments of protocols starting with anne6.

So, once more, suppose that the actual deal was not 012.345.6 but 245.013.6, that Anne says, as before, "My hand is one of 012, 034, 056, 135, 245, 246." In this scenario, Bill has not learnt that Cath's card is 6. How can Bill safely respond to Anne, and Anne to Bill, and so on? We will now systematically investigate all responses.

I do not know Cath's card Bill cannot admit that he doesn't know Cath's card, or, equivalently, that he doesn't know Anne's hand, because he would then be giving away that Anne's hand must be either 245 or 246. Cath would therefore learn that Anne has 2 and 4. Lost.

Please say something again, Anne He also cannot say *nothing*, or in more polite phrasing: "Please say something again, Anne." This is because Anne cannot respond to *that*: After Anne's announcement, Bill does not know whether Anne has 245 or 246. If Anne actually held 246 she could respond to Bill's request by saying, after all, that her hand is one of 012, 034, 056, 135, 246; i.e. she simply leaves out 245. But she holds 245, and no strict subset of $\{012, 034, 056, 135, 245, 246\}$ containing 245 is safe! This is easy to observe:

Bill's response has only heuristic and no informative content. Therefore, we can analyze Anne's second announcement as if it were made in the original epistemic state $(Rus, 245.013.6)$. Now such a subset cannot also contain 246; suppose 012 were left out, then if Cath held 5 she would learn that Anne held 4; suppose 034 were left out, then if Cath held 2, she would learn that Anne held 5; etc. But $\{012, 034, 056, 135, 245\}$ is also unsafe; e.g., if Cath had 0, she would learn that Anne has 5. So she can't say anything, apart from "Please say something again, Bill." But then we are back where we started: what should Bill say?

Two obvious replies of Bill to Anne have been outruled now. Bill cannot say that he does not know Cath's card yet, but he can also not try to hide that information by being non-committal. It seems we have run out of options. But this is far from the case: there are *many* others! After Anne says that she has one of 012, 034, 056, 135, 245, and 246, the domain consists of 24 card deals. These make up 23 b-equivalence classes: all are singleton, except $\{245.013.6, 246.013.5\}$. An announcement of Bill is interpreted as a union of b-classes, and as Bill is truthful, the b-class that contains the actual deal must always be included. But that means that any subset of these 23 classes that includes $\{245.013.6, 246.013.5\}$ denotes a possible reply of Bill to Anne. That makes 2^{22} replies to choose from. So far, we have ruled out two: the subset $\{245.013.6, 246.013.5\}$ corresponding to "I don't know your hand, Anne," and the subset of *all* b-classes, corresponding to "Please say something again, Anne." Note that we could rule out these replies by investigating any possible reply of Anne to that reply of Bill, and so on. There are therefore $2^{22} - 2$ remaining possible replies of Bill to investigate, and for each of those we have to consider all replies Anne can make to Bill's, and subsequent replies of Bill to Anne's second announcement, etc., ad infinitum. This appears rather intractable, if not undecidable at that...

Fortunately, we can systematically investigate all cases by further logical and combinatorial analysis. One important observation is, that we can assume a maximum number of announcements in a protocol. This is because all informative announcements restrict the domain, because the domain is finite, and because uninformative replies of the kind "please say something again" are meaningless if they occur (as above) at least twice in a row. And another important observation is that announcements must be safe: most of the b-classes are deals; for a given card (that Cath may hold), we must have enough deals to ensure safety, i.e., to ensure Cath's ignorance if she held that card. Now a set of two deals is only safe if Anne's and Bill's hands are disjoint, i.e., if it has the form $\{ijk.klm.n, klm.ijk.n\}$. But no two out of the 24 deals that we consider here have that property anyway. So if Bill's announcement includes one deal for some card q, it must contain at least two more deals where Cath holds q.

Now consider the following reply – it turns out this one will be safe, and typical for *all* other safe replies:

Cath has 5 or 6 Suppose Bill says "Cath has card 5 or card 6." This is true, and this is safe. Common knowledge of Cath's ignorance was already established after Anne's first announcement, and will obviously remain true for any union of b-classes that is a union of c-classes. The following visualize the models underlying the epistemic states before and after the response:

```
012.345.6 012.346.5 012.356.4 012.456.3
034.125.6 034.126.5                     034.156.2 034.256.1
                    056.123.4 056.124.3 056.134.2 056.234.1
135.024.6           135.026.4           135.046.2           135.246.0
245.013.6*                    245.016.3           245.036.1 245.136.0
          246.013.5*          246.015.3           246.035.1 246.135.0
```

becomes, after announcement of $K_b(5_c \vee 6_c)$:

```
012.345.6 012.346.5
034.125.6 034.126.5
135.024.6
245.013.6*
          246.013.5*
```

What can Anne say that is still safe? She cannot say: "Cath has card 6," because that would make it public that she knows that, which would eliminate the a-classes $\{012.345.6, 012.346.5\}$ and $\{034.125.6, 034.126.5\}$ where she does not know that, so the resulting model would be

```
135.024.6
245.013.6
```

so that Cath knows that Anne holds card 5 and is no longer ignorant. Formally we have that $Rus, 245.013.6 \models [\mathsf{anne6}][K_b(5_c \vee 6_c)][K_a 6_c]\neg\mathsf{cignorant}$ because $Rus|\mathsf{anne6}|K_b(5_c \vee 6_c)|K_a 6_c, 245.013.6 \models \neg\mathsf{cignorant}$ because $K_c 5_a$ holds in the last epistemic state. Is there anything else she can say in response? Yes: Anne can respond with "I do not have 135." We then get the model

```
012.345.6 012.346.5
034.125.6 034.126.5
245.013.6*
          246.013.5*
```

This announcement is safe, because it remains common knowledge that Cath is ignorant. What can Bill say in return, after that? Unfortunately, nothing informative: any restriction of either of the current c-classes makes them unsafe – for the given card deals, any c-class with fewer than three elements is unsafe. Right, so we are down to $2^{22} - 3$ remaining cases. What next?

Variations on 'Cath has 5 or 6' What deal can we remove from Bill's reply "Cath has 5 or 6" such that it remains safe? The actual b-class $\{245.013.6, 246.013.5\}$ must always be included (or Bill would be lying). Therefore, at least two deals where Cath holds 5 and at least two deals where Cath holds 6 have to be included as well, such that it remains common knowledge that she is ignorant (see above). There are only two other deals where Cath holds 5, that therefore both need to be included. There are three other deals where Cath holds 6, and any two out of these three keep Cath ignorant. For example, if Bill omits 135.024.6 ("My hand of cards is one of 345 346 125 126 024 013.") we get

```
012.345.6 012.346.5
034.125.6 034.126.5
245.013.6*
          246.013.5*
```

and Anne cannot respond informatively to Bill at all, nor can Bill if Anne were to ask Bill to respond once more. Any strict subset of a-classes is unsafe, as before. If Bill omits 012.345.6 instead, we get

```
          012.346.5
034.125.6 034.126.5
135.024.6
245.013.6*
          246.013.5*
```

and again Anne cannot respond. The case where Bill omits 034.125.6 instead, is similar. Down to $2^{22} - 6$; well...

Other replies We have now analyzed six possible replies, two of those were unsafe, four were safe but were leading nowhere. Now take any of those four, and consider adding any number of the remaining b-classes, i.e. any card deal where Cath does not hold 5 or 6. For example, taking the last epistemic model, we get, schematically

```
          012.346.5  +++
034.125.6 034.126.5  +++
                     ++
135.024.6            +++
245.013.6*           +++
          246.013.5* +++
```

where +++ and ++ denote possible other card deals included by Bill, and ++ specifically those where Anne holds 056. Very similar to the scenarios before, Anne cannot now say anything: the hands she will announce always must include her actual hand 245 and therefore as well 034 and 135, because any subset is unsafe, but that means she will have to carry along at least one deal where Cath holds 5 as well, namely 034.126.5. But therefore she will have to include enough other deals where Cath holds 5 so that the c-equivalence class for card 5 is safe, so therefore she must always include her hands 012 and 246 anyway. The only remaining hand is now 056. Suppose Bill had included, for example, 056.123.4 in his announcement, then he would have been obliged to include 012.356.4 and 135.026.4 as well, and because Anne's announcement includes hand 012 and 135 it therefore must include 056 as well. Suppose, instead, Bill had included

056.124.3 in his announcement, then ..., etc. So Anne cannot delete *a single* from her six hands, i.e., she cannot make an informative announcement. The other cases are just as similar to another one of the 'Cath has 5 or 6' variations that we have already discussed. Either Anne cannot respond at all, as here, or Anne may be able to reply to that she does not have 135, to which then Bill cannot respond. That was, after all, rather quick for $2^{22} - 6$ remaining cases!

We have now established the following. If the card deal is 245.013.6, then after Anne has said "I have one of 012, 034, 056, 135, 245, 246," in whatever way Bill responds to that, either Anne cannot respond informatively, or Anne can make an informative response to which Bill then cannot respond. Therefore, no *effective* protocol for card deal 245.013.6 starts with Anne saying that she has one of 012, 034, 056, 135, 245, 246. We assume that Anne, and Bill, take no risks: they are only willing to execute protocols that guarantee success, in the sense that, whatever one says, the other can make at least one safe reply to that which will bring a solution closer. Therefore, hand 245 is publicly known not to be Anne's actual hand. But that means that $\mathsf{anne5}; \mathsf{bill}$ and $\mathsf{anne5}; \mathsf{bill}$ are 'essentially the same'. This is a minor result, but a result all the same, and we have answered one of the remaining riddles concerning Russian Cards.

We now close this contribution with some additional observations. First, this: We started out with assuming that the deal was 012.345.6, not 245.013.6, and for *that* deal, $\mathsf{anne5}$ was contained in $\mathsf{anne6}$ *anyway.* So couldn't we have saved us all this trouble? A different way to describe the result, that makes clearer that it is indeed a result, is to turn matters around: Suppose we *had* established that $\mathsf{anne6}$ starts a sequence of announcements that provides a solution for deal 245.013.6. Then Anne could have executed the same underlying protocol if her hand had been 012 and 'with 012 in the role of 245', as in deal 012.345.6. That would have resulted in, for example, Anne saying: "My hand is one of 012, 026, 034, 135, 146, 245." None of the solutions listed in [vD03] starts with Anne announcing a subset of that. So, indeed, that would have been a new solution to the problem.

What is the meaning of 'essentially the same' in the above? Is using the word 'essential' not just a clever trick to be excused from formal precision? We observed that an announcement of φ in this setting should be interpreted as $K_n\varphi \wedge [K_n\varphi]C_N\mathsf{cignorant}$. Given that interpretation, the models $Rus|\mathsf{anne5}$ and $Rus|\mathsf{anne6}$ were different. Apparently, we need to incorporate even more pragmatic information into the meaning of φ, or in other words, φ should be interpreted as $K_n\varphi \wedge [K_n\varphi](C_N\mathsf{cignorant} \wedge \psi)$, for some ψ or other. Subject to *that* interpretation, $Rus|\mathsf{anne5}$ and $Rus|\mathsf{anne6}$ are, apparently, the same. What is ψ?

We constructed such a ψ above. Let ψ_{ba} quantify over all informative responses of Bill to Anne's announcement $\mathsf{anne6}$, that is over all descriptions of subsets of the 24 possible b-hands that include actual b-class $\{245, 246\}$. In other words, those formulas contain a part $245_b \vee 246_b$, and various other parts ijk_b. Let ψ_{aba} quantify over all informative reponses of Anne to that, and ψ_{baba} over all informative responses of Bill to that. Abbreviate $C_{abc}(\mathsf{bknowsas} \wedge \mathsf{aknowsbs})$ as solved. Then we have shown that the following formula is *false* in state 245.013.6 of model $Rus|\mathsf{anne6}$:

$$\begin{array}{l} \bigwedge_{\psi_{ba}} [K_b \psi_{ba}](C_{abc}\mathsf{cignorant} \wedge \neg\mathsf{solved} \rightarrow \\ \qquad \bigwedge_{\psi_{aba}} [K_a \psi_{aba}](C_{abc}\mathsf{cignorant} \wedge \neg\mathsf{solved} \rightarrow \\ \qquad\qquad \bigvee_{\psi_{baba}} \langle K_b \psi_{baba} \rangle C_{abc}\mathsf{cignorant})) \end{array}$$

Substitute public knowledge of that formula for ψ in $K_n\varphi \wedge [K_n\varphi](C_N\mathsf{cignorant} \wedge \psi)$, and, in the last, a for n, and $\{a, b, c\}$ for N, and either $\mathsf{anne5}$ or $\mathsf{anne6}$ for $K_a\varphi$, then, indeed, the result of Anne announcing either one or the other in epistemic state $(Rus, 012.345.6)$ is the same. We omit details.

In other words, Anne only says something, if it is safe and – unless the problem is solved – if Bill has at least one safe response to that, to which he knows that – unless the problem is now solved – Anne will be able to respond safely, and so on, until the problem is solved. In our example 'depth four' sufficed to uncover a non-solution. In general, the depth required is finite and a function of the card deal. A more elegant formulation – although not strictly necessary – is to be expected in a more expressive logic, namely the logic of public announcements with arbitrary iteration of announcements [MM04]: the 'Kleene star' operation on announcements.

There are some other lose threads as well to wind up; e.g., why is it required that it is *public knowledge* that Anne and Bill know each others' cards: $C_{abc}(\mathsf{aknowsbs} \wedge \mathsf{bknowsas})$? One can show that it is sufficient and necessary that just Anne and Bill have common knowledge of that: $C_{ab}(\mathsf{aknowsbs} \wedge \mathsf{bknowsas})$. Could it be the case, that it is then publicly known as well? Somewhat surprisingly, *before* any announcements are made, the frame underlying that epistemic state satisfies the schema $C_{ab}\varphi \rightarrow C_{abc}\varphi$, but this is no longer the case *after* announcements have been made. Maybe it still is, for that instance of φ.

We hope to continue this investigation in those directions.

References

[BM04] A. Baltag and L.S. Moss. Logics for epistemic programs. *Synthese*, 139:165–224, 2004. Knowledge, Rationality & Action 1–60.

[MM04] J.S. Miller and L.S. Moss. The undecidability of iterated modal relativization. Manuscript, 2004.

[vD03] H.P. van Ditmarsch. The russian cards problem. *Studia Logica*, 75:31–62, 2003.