Explorations and computations in bidirectional intuitionistic propositional logic

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Abstract

This paper investigates the semantics of the intuitionistic propositional logic (**IPL**) extended with subtraction, also known as the Heyting-Brouwer logic or **biIPL**. It introduces and extends some basic concepts and theorems in the Kripke semantics of intuitionistic propositional logic to obtain new results on the exact models of fragments in the bidirectional case. The paper also includes results of the computations in fragments of **biIPL**, based on some of these exact models.

1 Introduction

Apart from it's philosophical interest, intuitionistic logic has attracted researchers for it's information oriented flavor. The underlying idea that formulas describe the way an idealised mathematician acquires new mathematical knowledge, can be generalised for example to include the description of the changes in the information state of a database.

The intuitionistic logic especially describes such changes in the 'essential' part of our knowledge: once such an 'essential' truth is established it will be part of all following information states.

Even if one respects the expanding nature of intuitionistic logic, it could be interesting to study a logic which can also take into account *previous* information states and would allow moving forward *and* backward between such states.

One such an extension of the intuitionistic propositional logic is the propositional Heyting-Brouwer logic, obtained by extending the intuitionistic propositional logic **IPL** with a new connective dual to the implication. The rule for this *subtraction* operator \ is:

$$A \backslash B \vdash C \quad \Leftrightarrow \quad A \vdash B \vee C$$

The semantic definition of subtraction in Kripke models of **IPL** is:

$$m \models A \backslash B \quad \Leftrightarrow \quad \exists l \leq m \ (l \models A \text{ and } l \not\models B)$$

The logic **biIPL** was introduced in [19] and has been studied, at least to some extend, in [4], [5], [9], [20], [21], [22] and [24]. In these papers several names can be found for the logic (Heyting-Brouwer logic, logic with coimplication, bidirectional intuitionistic logic, dual intuitionistic logic) and for subtraction (coimplication, pseudo-difference).

Compared to the usual semantics of **IPL** the semantics of **biIPL** is bidirectional. Because of the semantics of the subtraction operator, the truth of a formula in a certain world m no longer only depends on the situation in m and the worlds *above* m, but may also depend on the worlds *below* m.

This paper investigates the consequences of the bidirectional semantics of **biIPL** for some basic notions that have been introduced in the semantics of **IPL** in the research on exact models and computations of fragments (see [10], [11]).

2 Preliminaries

The language of the propositional logic in this paper uses connectives in the set $\{\wedge, \vee, \rightarrow, \backslash\}$. The rules for **biIPL** are the usual ones for **IPL** plus the $\backslash$-rule

$$A \backslash B \vdash C \quad \Leftrightarrow \quad A \vdash B \vee C$$

We will assume $\neg A$ is defined as $A \rightarrow \bot$ and that $A \leftrightarrow B$ is a shorthand for $(A \rightarrow B) \wedge (B \rightarrow A)$. Similarly we introduce a dual for the negation, $\lrcorner A = \top \backslash A$. To reduce the number of parentheses in formulas we will assume the following order of preference: $\lrcorner \; = \; \neg \; > \; \wedge \; > \; \vee \; > \; \rightarrow = \; \backslash$.

For example the formula $(((A \wedge B) \rightarrow (C \vee \neg(D))) \backslash (\lrcorner(\neg(E))))$ will be written as $(A \wedge B \rightarrow C \vee \neg D) \backslash \lrcorner \neg E$.

In a *fragment* of **biIPL** the number of atoms and connectives in the language is restricted. For the derivability relationship in such a fragment we use the derivability in **biIPL**. The notation for these fragments consists of the list of connectives in the restricted language, written between square brackets, and a superscript number at the end indicating the number of atoms in the fragment. Hence $[\wedge, \backslash]^2$ consists of all formulas build from two atoms with conjunction and subtraction alone.

The reader is assumed to be acquainted with the Kripke semantics for intuitionistic propositional logic (see for example [6] for details). In the sequel the frames $\langle W, \leq \rangle$, on which our Kripke models $\langle W, \leq, atom \rangle$ are based, will be (usually finite) partial ordered sets (not necessarily rooted). The function *atom* defines for each world in $m \in W$ the atomic formulas p in the language true in m, so $atom(m) = \{p \text{ atomic} \mid m \models p\}$.

In the Kripke semantics for intuitionistic propositional logic the important constraint on this function is that $[\![p]\!] = \{m \in W \mid m \models p\} = \{m \in W \mid p \in atom(m)\}$ should be an upward closed subset (a *cone*) of W. I.e. $p \in [\![m]\!]$ and $m \leq l$ (for $l \in W$) then $p \in atom(l)$.

As the focus of this paper is mainly on the semantics of **biIPL** we usualy will not refer to a specific proof system for this extension of **IPL**. If the reader would be in need of such a system, the following set of rules will do. The format of these rules has been chosen to clarify the duality principle in **biIPL** (see fact 28).

Let Γ and Δ be finite sets of formulas. A sequent $\Gamma \vdash \Delta$ can informally be read as 'from the assumptions in Γ one can derive one of the conclusions of Δ'.

In the sequel we will, as usual, often write Γ, A for $\Gamma \cap \{A\}$ if A is a formula and Γ a set of formulas.

The *structural* rules for this calculus are:

(Ax)	$\Gamma \cap \Delta \neq \emptyset$	$\Rightarrow$	$\Gamma \vdash \Delta$
(Weak)	$\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ and $\Gamma' \vdash \Delta'$	$\Rightarrow$	$\Gamma \vdash \Delta$
(Cut)	$\Gamma \vdash A, \Delta$ and $\Gamma, A \vdash \Delta$	$\Rightarrow$	$\Gamma \vdash \Delta$

Let A and B be formulas. For each of the connectives in $\{\wedge, \vee, \rightarrow, \backslash\}$ we now introduce a rule:

$$
\begin{array}{llllcllll}
(\wedge) & \Gamma, A \wedge B & \vdash & \Delta & \Leftrightarrow & \Gamma, A, B & \vdash & \Delta & \\
(\vee) & \Gamma & \vdash & A \vee B, \Delta & \Leftrightarrow & \Gamma & \vdash & A, B, \Delta & \\
(\rightarrow) & \Gamma & \vdash & A \rightarrow B, \Delta & \Leftrightarrow & \Gamma, A & \vdash & B, \Delta & (\Delta = \emptyset) \\
(\backslash) & \Gamma, A \backslash B & \vdash & \Delta & \Leftrightarrow & \Gamma, A & \vdash & B, \Delta & (\Gamma = \emptyset)
\end{array}
$$

Leaving out the $\backslash$-rule, the above proof system can be proved to be equivalent to other, more usual, sequent calculi for **IPL**. The conditions on the $\rightarrow$-rule and the $\backslash$-rule are essential for **biIPL**. Lifting these would result in a calculus for the classical propositional logic (**CPL**).

The above rules, akin to those of Ketonen in [18], suggest that one could try to add new connectives like:

$$
\begin{array}{lllclll}
\Gamma, A \circ B & \vdash & \Delta & \Leftrightarrow & \Gamma & \vdash & A, B, \Delta \\
\Gamma & \vdash & A \circ B, \Delta & \Leftrightarrow & \Gamma, A, B & \vdash & \Delta.
\end{array}
$$

These connectives can however be defined in **biIPL** as:

$$
\begin{array}{lcl}
\top \backslash (A \vee B) \vdash \Delta & \Leftrightarrow & \vdash A, B, \Delta \\
\Gamma \vdash \neg(A \wedge B) & \Leftrightarrow & \Gamma, A, B \vdash
\end{array}
$$

(again, lifting the conditions on Δ or Γ in these rules would yield **CPL**).

As, obviously, $\Gamma \vdash B \rightarrow A \quad \Leftrightarrow \quad \Gamma, B \vdash A$ (and likewise for the $\backslash$-rule), the logic **biIPL** is in a sense complete for this type of rules defining connectives.

3 Basic semantics

This section introduces some basic notions for the semantics of **biIPL**. Some of these notions, like *situations*, *semantic types* and *exact Kripke models* may not be as standard as for example *bisimulation.* Neither is our approach in this section, based on fragments of **biIPL** with restricted nesting of both the implication and the subtraction the most common, and maybe not even the most economical, way to present the semantics of a propositional logic.

The non-standard approach in this section offers us the opportunity to show how these concepts, that have been developed for **IPL** and modal logics in previous research in co-operation with Dick de Jongh, are related and how they can be extended to cover the semantics of **biIPL**.

The following lemma, stating the soundness of the Kripke semantics of **biIpl** as described in the introduction section, will be the starting point of our investigation into the semantics of **biIPL**.

Lemma 1 **biIPL**, *defined as* **IPL** *extended with the* $\backslash$*-rule,*

$$A\backslash B \vdash C \quad \Leftrightarrow \quad A \vdash B \vee C,$$

is sound for the usual Kripke models of **IPL**, *where* $m \models A\backslash B$ *is defined as:*

$$m \models A\backslash B \quad \Leftrightarrow \quad \exists l \leq m.l \models A \textit{ and } m \not\models B$$

Proof. As we already know **IPL** is sound (and complete) for the Kripke models of **IPL**, basically, what has to be proved is that the $\backslash$-rule is valid with the given interpretation of $A\backslash B$. Hence $A\backslash B \models C \quad \Leftrightarrow \quad A \models B \vee C$.

Let $A\backslash B \models C$ and $m \models A$. If $m \not\models B$ then $m \models A\backslash B$ and hence by assumption $m \models C$. So in this case $m \models B \vee C$. If on the other hand $m \models B$ then obviously $m \models B \vee C$. Hence, $A\backslash B \models C \Rightarrow A \models B \vee C$.

For the other direction, let $A \models B \vee C$ and $m \models A\backslash B$. By definition, there is a $l \leq m$ such that $l \models A$ and $l \not\models B$. By assumption $l \models C$. As a consequence, $m \models C$ and hence $m \models B \vee C$. Which shows $A \models B \vee C \Rightarrow A\backslash B \models C$. $\dashv$

The proof of the completeness theorem is postponed until after the introduction of situations.

The following lemma introduces the notion of restricted nesting of implication and subtraction, the language of formulas with this nesting restricted to a certain number k, and the theory in this restricted language for a node in a Kripke model.

Definition 2 *Let* $\mathcal{L}$ *be the language of* **biIPL** *and* m *a node in a Kripke model* M. *Define*

- $\delta(A) = 0$ *if* A *atomic*
- $\delta(A \circ B) = \begin{cases} max\{\delta(A), \delta(B)\} & \textit{if} \circ \in \{\wedge, \vee\} \\ max\{\delta(A), \delta(B)\} + 1 & \textit{if} \circ \in \{\rightarrow, \backslash\} \end{cases}$

$\mathcal{L}_k = \{A \in \mathcal{L} \mid \delta(A) \leq k\}$

$Th_k(m) = \{A \in \mathcal{L}_k \mid m \models A\}$

In the sequel of this section we will assume that the fragments of **biIPL** are based on a finite number of atoms, n. To be more precise, we would have to show this n in our notation and write for example $\mathcal{L}_k^n$ and $Th_k^n(m)$ in stead of

$\mathcal{L}_k$ and $Th_k(m)$. For simplicity and readability however we will omit this n if there is no risk of confusion.

Note that, as a consequence of our definitions of $\neg$ and $\lrcorner$, we have $\delta(\neg A) = \delta(\lrcorner A) = \delta(A) + 1$.

Lemma 3 *For each finite n and finite k, the the number of equivalence classes in the fragment $\mathcal{L}_k^n$ of* **biIPL** *is finite.*

Proof. The proof proceeds by induction on k. Note that $\mathcal{L}_0^n = [\wedge, \vee]^n$, which has a finite number of equivalence classes.

Next, observe that each equivalence class in $\mathcal{L}_{k+1}^n$ is either a class in $\mathcal{L}_k^n$, in $\{A \to B \mid A, B \in \mathcal{L}_k^n\}$, or in $\{A \backslash B \mid A, B \in \mathcal{L}_k^n\}$. Clearly each of these sets has a finite number of equivalence classes (by the induction hypothesis). $\dashv$

3.1 Situations

In [14] Dick de Jongh proved that for each node m in a finite Kripke model M one can find two formulas, say $A(m)$ and $B(m)$, that in a sense define the semantic situation of m precisely. In fact if $l \models A(m)$ then the submodel generated by l, the cone $\uparrow l$, is bisimular to a generated submodel of $\uparrow m$, the cone generated by m. Dually, if $l \not\models B(m)$, $\uparrow m$ is bisimular to a generated submodel of $\uparrow l$. So if $l \models A(m)$ and $l \not\models B(m)$, the nodes m and l are bisimular.

Here we first generalise this idea, introducing pairs $\langle A, B\rangle$ as situations (where A will be true and B false) and then use a specific kind of situations to prove the completeness theorem for **biIPL**. In subsection 3.2 we will prove a special version of the above mentioned theorem (also known as the De Jongh/Jankov/Fine theorem) in the context of **biIPL** by relating the so called *maximal* situations to *semantic types.*

Definition 4 *Define*

- *A situation (in $\mathcal{L}$) is a tuple of formulas, $\langle A, B\rangle$ in $\mathcal{L}$.*
- *A situation $\langle A, B\rangle$ is* consistent *if $A \nvdash B$.*
- *A situation $\langle A, B\rangle$ is* prime *if for all C (in $\mathcal{L}$) $A \nvdash C \quad \Leftrightarrow \quad A, C \vdash B$*
- *A situation $\langle A, B\rangle$ is called* maximal *if for each formula C (in $\mathcal{L}$) it is true that $A \nvdash C \quad \Leftrightarrow \quad C \vdash B$.*

The following definitions link the situations defined above to Kripke semantics.

Definition 5 *If M is a Kripke model and m a node in M define*

- $m \models \langle A, B\rangle \quad \Leftrightarrow \quad m \models A$ *and* $m \not\models B$.
- $[\![\langle A, B\rangle]\!]_M = \{m \mid m \models \langle A, B\rangle\}$

- $\langle A, B\rangle \vdash \langle C, D\rangle \quad \Leftrightarrow \quad [\![\langle A, B\rangle]\!]_M \subseteq [\![\langle C, D\rangle]\!]_M$
- $A_{\mathcal{L}_k}(m) = \bigwedge\{C \in \mathcal{L}_k \mid m \models C\}$
- $B_{\mathcal{L}_k}(m) = \bigvee\{C \in \mathcal{L}_k \mid m \not\models C\}$
- $\sigma_{\mathcal{L}_k}(m) = \langle A_{\mathcal{L}_k}(m), B_{\mathcal{L}_k}(m)\rangle$ *is called the k-situation of m (in $\mathcal{L}$).*

If there is no risk of confusion, we will omit mentioning $\mathcal{L}$ or the Kripke model M and for example simply write $A_k(m), \sigma_k(m)$ and $[\![\langle A, B\rangle]\!]$.

Lemma 6 *The k-situation of a node m in a Kripke model M is a maximal situation in $\mathcal{L}_k$.*

Proof. For each $C \in \mathcal{L}_k$ either $m \models C$ and $A_k(m) \vdash C$ or $m \not\models C$ and $C \vdash B_k(m)$. So we have indeed for each $C \in \mathcal{L}_k$: $A_k(m) \nvdash C \quad \Leftrightarrow \quad C \vdash B_k(m)$.⊣

The following facts are easy to prove.

Facts 7

1. *If $[\![\langle A, B\rangle]\!]_M \neq \emptyset$ for some Kripke model M then $\langle A, B\rangle$ is consistent.*
2. *Prime situation are consistent and maximal situations are prime.*
3. *If m a node in a Kripke model M, then $\sigma_k(m)$ is a maximal situation in $\mathcal{L}_k$.*
4. *A prime situation $\langle A, B\rangle$ is maximal if $A \to B \vdash B$.*
5. *If $\langle A, B\rangle$ is prime in $\mathcal{L}$ and $A \to B$ is a $\mathcal{L}$-formula, then $\langle A, A \to B\rangle$ is maximal in $\mathcal{L}$.*
6. *If $\langle A, B\rangle$ is prime in $\mathcal{L}$ and $C, D, C \vee D$ are formulas in $\mathcal{L}$, then $A \vdash C \vee D \quad \Leftrightarrow \quad A \vdash C$ or $A \vdash D$*
7. $\langle A, B\rangle \vdash \langle C, D\rangle \quad \Leftrightarrow \quad A \vdash B \vee C$ *and* $A \wedge D \vdash B$
8. $Th_k(m) = \{C \in \mathcal{L}_k \mid A_k(m) \vdash C\}$
9. $B_k(m) = \bigvee\{C \in \mathcal{L}_k \mid A_k(m) \nvdash C\}$

Note that, as $\mathcal{L}_k^n$ has finitely many equivalence classes, there are, up to equivalence in **biIPL**, finitely many k-situations in $\mathcal{L}_k^n$. We can define a partial ordering of these k-types.

Definition 8 *Define $\sigma_k(m) \leq \sigma_k(l) \quad \Leftrightarrow \quad A_k(l) \vdash A_k(m)$.*

The following lemma shows that we could also have used the $B_k(m)$ part of the $\sigma_k(m)$ for this definition.

Lemma 9 *Let m and l be nodes in a Kripke model M, then*

$$\sigma_k(m) \leq \sigma_k(l) \quad \Leftrightarrow \quad B_k(l) \vdash B_k(m)$$

Proof. Let $\sigma_k(m) \leq \sigma_k(l)$. By definition $A_k(l) \vdash A_k(m)$ and as $A_k(l) \nvdash B_k(l)$ also $A_k(m) \nvdash B_k(l)$. As $\sigma_k(m)$ is maximal, this proves $B_k(l) \vdash B_k(m)$.

For the other direction, let $B_k(l) \vdash B_k(m)$. So $A_k(m) \nvdash B_k(l)$ and, by maximality of $\sigma_k(l)$, $A_k(l) \vdash A_k(m)$. Which proves $\sigma_k(m) \leq \sigma_k(l)$. ⊣

The following fact is a simple consequence of lemma 9

Fact 10 *Let $\langle A, B\rangle$ be a maximal situation in $\mathcal{L}_k$ and m a node in a Kripke model M. Then $m \models A$ and $m \not\models B \quad \Leftrightarrow \quad \sigma_k(m) = \langle A, B\rangle$*

We will now prove a completeness theorem for fragments of **biIPL** with restricted nesting of $\to$ and $\backslash$, using the partial ordering of all the k-situations in $\mathcal{L}_k^n$ as a finite Kripke model.

This proof is akin to the construction of finite Henkin-Kripke models. The following lemma will play the role normally played by the Lindenbaum Lemma.

Let us fix n for $\mathcal{L}_k^n$ and in the sequel simply write $\mathcal{L}_k$.

Lemma 11 *For each consistent situation $\langle A, B\rangle$ in $\mathcal{L}_k$, there exists a maximal situation $\langle A', B'\rangle$ in $\mathcal{L}_k$ such that $A' \vdash A$ and $B \vdash B'$.*

Proof. The proof is in fact a construction similar to that in the usual proof of the Lindenbaum Lemma.

Enumerate all equivalence classes in $\mathcal{L}_k$ and let each be represented by a formula C_i in $\mathcal{L}_k$. Let $A_0 = A$ and $B_0 = B$ and define:

$$\langle A_{i+1}, B_{i+1}\rangle = \begin{cases} \langle A_i \wedge C_i, B_i\rangle & \text{if } A_i, C_i \nvdash B_i \\ \langle A_i, B_i \vee C_i\rangle & \text{otherwise} \end{cases}$$

After a finite number of steps, say l, the list of C_i's is exhausted and define $\langle A', B'\rangle = \langle A_l, B_l\rangle$.

From this definition of $\langle A', B'\rangle$ it easily follows that $\langle A', B'\rangle$ is a maximal situation and $A' \vdash A$ and $B \vdash B'$. ⊣

Definition 12 *Define $M_{\mathcal{L}_k}$ as the partial ordering of all the maximal situations in $\mathcal{L}_k$ with the valuation $atom(\langle A, B\rangle) = \{p \text{ atomic } \mid A \vdash p\}$*

On can easily check that $M_{\mathcal{L}_k}$ is indeed a Kripke model (for **IPL** and hence for **biIPL**).

The next lemma shows that each maximal situation in $\mathcal{L}_k$ reflects precisely its situation in $M_{\mathcal{L}_k}$.

Lemma 13 *If $\langle A, B\rangle$ is a maximal situation in $M_{\mathcal{L}_k}$ then $\sigma_k(\langle A, B\rangle) = \langle A, B\rangle$.*

Proof. We will prove that for each $\langle A, B\rangle$ in $M_{\mathcal{L}_k}$ and each $C \in \mathcal{L}_k$:

$$\langle A, B\rangle \models C \quad \Leftrightarrow \quad A \vdash C \quad \Leftrightarrow \quad C \nvdash B$$

Note that the last part of this statement is a simple consequence of the definition of a maximal situation.

As an immediate consequence we have that $\langle A, B\rangle \models A$ and $\langle A, B\rangle \not\models B$. By fact 10 this proves that $\sigma_k(\langle A, B\rangle) = \langle A, B\rangle$.

The proof that $\langle A, B\rangle \models C \quad \Leftrightarrow \quad A \vdash C$ proceeds by induction on the complexity of the formula C. If C is atomic, the statement is true by definition.

$\langle A, B\rangle \models C \wedge D$ iff both $\langle A, B\rangle \models C$ and $\langle A, B\rangle \models D$. By induction hypothesis this is true iff $A \vdash C$ and $A \vdash D$. Which is equivalent to $A \vdash C \wedge D$.

For the case of $C \vee D$ the prove is similar, using the fact that maximal situations are prime and hence $A \vdash C \vee D \quad \Leftrightarrow \quad A \vdash C$ and $A \vdash D$.

Now let $\langle A, B\rangle \models C \to D$ and assume $A \nvdash C \to D$. This would imply that $A \wedge C \nvdash D$ and hence $\langle A \wedge C, D\rangle$ is a consistent situation. By lemma 11 there would be a maximal $\langle A', B'\rangle$ such that $A' \vdash A \wedge C$ and $D \vdash B'$. So by induction hypothesis $\langle A', B'\rangle \models C$ and $\langle A', B'\rangle \not\models D$. But from $A' \vdash A$ one infers that $\langle A, B\rangle \leq \langle A', B'\rangle$, contradicting $\langle A, B\rangle \models C \to D$. Which proves $\langle A, B\rangle \models C \to D$ implies $A \vdash C \to D$.

For the other direction, assume $A \vdash C \to D$. Let $\langle A, B\rangle \leq \langle A', B'\rangle$. If $\langle A', B'\rangle \models C$, then by induction hypothesis $A' \vdash C$. As also $A' \vdash A$ (because $\langle A, B\rangle \leq \langle A', B'\rangle$) one may conclude $A' \vdash D$ and hence $\langle A', B'\rangle \models D$ by the induction hypothesis. Which proves $\langle A, B\rangle \models C \to D$.

Let $\langle A, B\rangle \models C \backslash D$. So, for some maximal situation $\langle A', B'\rangle \leq \langle A, B\rangle$ (hence where $A \vdash A'$) we have $\langle A', B'\rangle \models C$ and $\langle A', B'\rangle \not\models D$. In other words: $A' \vdash C$ and $D \vdash B'$

Assume $A \nvdash C \backslash D$, then $C \backslash D \vdash B$ and hence $C \vdash D \vee B$. Now we would have $A' \vdash D \vee B$ and hence, as $D \vdash B'$, $A' \vdash B$. But as $A \vdash A'$ this would yield $A \vdash B$, a contradiction. Which proves $A \vdash C \backslash D$.

For the other direction, let $A \vdash C \backslash D$. As a consequence $C \backslash D \nvdash B$ and hence $C \nvdash D \vee B$. By lemma 11 there is a maximal $\langle A', B'\rangle$ such that $A' \vdash C$ and $D \vee B \vdash B'$. From $B \vdash B'$ infer that $\langle A', B'\rangle \leq \langle A, B\rangle$. Using the induction hypothesis one easily proves $\langle A', B'\rangle \models C$ and $\langle A', B'\rangle \not\models D$. Which proves $\langle A, B\rangle \models C \backslash D$. $\dashv$

As a simple consequence of lemma 13 and the construction of the model $M_{\mathcal{L}_k}$ each maximal situation in a fragment of **biIPL** with a finite number of atoms and restricted nesting of both implication and subtraction is the k-situation of a node in a (finite) Kripke model.

In combination with lemma 11 this almost immediately provides us with a proof of a completeness theorem for **biIPL**.

Theorem 14 *If T be a finite set of formulas in* **biIPL** *and A a formula in* **biIPL** *such that $T \nvdash A$, then for some node m in a finite Kripke model M it is true that $m \models T$ and $m \not\models A$.*

Proof. If $T \nvdash A$, there is a $\mathcal{L}^n_k$ such that both $\bigwedge T$ and A are formulas of $\mathcal{L}^n_k$. In this fragment $\langle \bigwedge T, A\rangle$ is a consistent situation. According to lemma 11 there is a maximal situation $\langle A', B'\rangle$ covering the situation $\langle \bigwedge T, A\rangle$. I.e. $A' \vdash \bigwedge T$ and $A \vdash B'$. The node $\langle A', B'\rangle$ in the model $M_{\mathcal{L}_k}$ for the fragment $\mathcal{L}^n_k$ is the required node m. ⊣

3.2 Semantic types

In [11] the notion of a semantic type was introduced (see [10] for more details on semantic types in **IPL**).

Definition 15 *Define the* semantic k-type of m, $t_k(m)$ *as*

$$t_k(m) = \begin{cases} atom(m) & \textit{if } k = 0 \\ \langle t_{k-1}(m), \{t_{k-1}(l) \mid m \leq l\}, \{t_{k-1}(l) \mid l \leq m\}\rangle & \textit{if } k > 0 \end{cases}$$

If t is a semantic k-type, then $\tau_0(t), \tau_1(t)$, and $\tau_2(t)$ are defined by stipulating $t = \langle \tau_0(t), \tau_1(t), \tau_2(t)\rangle$

As a consequence of this definition, if $m \leq l$ and $k > 0$ then both $\tau_1(t_k(l)) \subseteq \tau_1(t_k(m))$ and $\tau_2(t_k(m)) \subseteq \tau_2(t_k(l))$.

Lemma 16 *If t is a semantic k-type and $k > 0$ then $\tau_1(t) \cap \tau_2(t) = \{\tau_0(t)\}$*

Proof. Let $t = t_1(m)$ then $\forall l \geq m.atom(m) \subseteq atom(l)$ and $\forall l \leq m.atom(l) \subseteq atom(m)$. So $\{atom(l) \mid l \geq m\} \cap \{atom(l) \mid l \leq m\} = \{atom(m)\}$. Which proves the lemma for $k = 1$.

Now assume the lemma to hold for k. By definition it is true that $t_k(m) \in \tau_1(t_{k+1}(m)) \cap \tau_2(t_{k+1}(m))$. If $t = t_k(l)$ for some l and $t \in \tau_1(t_{k+1}(m)) \cap \tau_2(t_{k+1}(m))$ then there is a $l' \geq m$ such that $t_k(l') = t$ and a $l'' \leq m$ such that $t_k(l'') = t$.

So $\tau_1(t) = \tau_1(t_k(l')) \subseteq \tau_1(t_k(m)) \subseteq \tau_1(t_k(l'')) = \tau_1(t)$ and we may conclude that $\tau_1(t) = \tau_1(t_k(m))$. Similarly one proves $\tau_2(t) = \tau_2(t_k(m))$. Applying the induction hypothesis now yields $\tau_0(t_k(m)) = \tau_0(t)$. Hence $t = t_k(m)$ and $\tau_1(t_{k+1}(m)) \cap \tau_2(t_{k+1}(m) = \{t_k(m)\} = \{\tau_0(t_{k+1}(m))\}$. ⊣

Corollary 17 *If t_1 and t_2 are semantic k-types and $\tau_1(t_1) = \tau_1(t_2)$ and $\tau_2(t_1) = \tau_2(t_2)$ then $t_1 = t_2$.*

The corollary above justifies the following definition.

Definition 18 *Let t_1 and t_2 be semantic k-types. Define*

$$t_1 \leq t_2 \quad \Leftrightarrow \quad \begin{cases} t_1 \subseteq t_2 & \textit{if } k = 0 \\ \tau_1(t_2) \subseteq \tau_1(t_1) \textit{ and } \tau_2(t_1) \subseteq \tau_2(t_2) & \textit{otherwise} \end{cases}$$

One can easily check that the defined order on the semantic k-types respects the order in the Kripke models. That is, if $m \leq l$ then $t_k(m) \leq t_k(l)$. This fact is used in the following lemma.

Lemma 19 *Let m and l be nodes in a Kripke model M, then:*

$$t_{k+1}(m) \leq t_{k+1}(l) \quad \Rightarrow \quad t_k(m) \leq t_k(l)$$

Proof. Using definition 18 and the above mentioned fact, $t \in \tau_1(t_{k+1})$ implies $t \leq t_k(m)$. Similarly, $t \in \tau_2(t_{k+1}) \quad \Rightarrow \quad t \geq t_k(m)$. So, if $t_{k+1}(m) \leq t_{k+1}(l)$ then $\tau_1(t_{k+1}(l) \subseteq \tau_1(t_{k+1}(m)$. The later implies $t_k(l) \in \tau_1(t_{k+1}(m)$ and hence $t_k(m) \leq t_k(l)$. ⊣

Corollary 20 *If m and l nodes in a Kripke model M then $t_k(m) \leq t_k(l)$ implies $atom(m) \subseteq atom(l)$*

Proof. If $k = 0$ then the statement is trivial. So let $t_{k+1}(m) \leq t_{k+1}(l)$. According to the lemma above this implies $t_k(m) \leq t_k(l)$, which by induction hypothesis implies $atom(m) \subseteq atom(l)$. ⊣

As a consequence, the partial order of a set of k-types is a Kripke model if one defines $atom(t_k(m)) = atom(m)$.

Theorem 21 *If m and l nodes in a Kripke model M then the following are equivalent:*

1. $Th_k(m) \subseteq Th_k(l)$
2. $\sigma_k(m) \leq \sigma_k(l)$
3. $t_k(m) \leq t_k(l)$

Proof. Note that for $k = 0$ the theorem is trivial. By induction on k we will proof the general case. So, let $k > 0$ and assume the theorem to be true for j-theories, j-situations and j-types with $j < k$.

(1 $\Rightarrow$ 2) $A_k(m) \in Th_k(m) \subseteq Th_k(l)$ implies $A_k(l) \vdash A_k(m)$ which by definition implies $\sigma_k(m) \leq \sigma_k(l)$.

(2 $\Rightarrow$ 3) Let $t \in \tau_1(t_k(l))$, then for some $w \geq l$ it will be true that $t_{k-1}(w) = t$. As a consequence, $l \not\models A_{k-1}(w) \rightarrow B_{k-1}(w)$, and, as $\sigma_k(l)$ is maximal, one may conclude that $A_{k-1}(w) \rightarrow B_{k-1}(w) \vdash B_k(l)$. As we assume $\sigma_k(m) \leq \sigma_k(l)$, $B_k(l) \vdash B_k(m)$ and hence $A_{k-1}(w) \rightarrow B_{k-1}(w) \vdash B_k(m)$. Which implies $m \not\models A_{k-1}(w) \rightarrow B_{k-1}(w)$. So for some $w' \geq m$ it will be true that $w' \models A_{k-1}(w)$ and $w' \not\models B_{k-1}(w)$. By fact 10 this implies $\sigma_{k-1}(w) = \sigma_{k-1}(w')$ and by the induction hypothesis this implies $t_{k-1}(w) = t_{k-1}(w') \in \tau_1(t_k(l))$. Which proves $\tau_1(t_k(m)) \subseteq \tau_1(t_k(l))$.

Let $t \in \tau_2(t_k(m))$. For some $w \leq m$ it will be true that $t_{k-1}(w) = t$. As a consequence, $m \models A_{k-1}(w) \backslash B_{k-1}(w)$, and $A_k(m) \vdash A_{k-1}(w) \backslash B_{k-1}(w)$. From the assumption that $\sigma_k(m) \leq \sigma_k(l)$ it follows that $A_k(l) \vdash A_k(m)$. Which implies that $A_k(l) \vdash A_{k-1}(w) \backslash B_{k-1}(w)$.

By definition then $l \models A_{k-1}(w) \backslash B_{k-1}(w)$. So for some $w' \leq l$ it will be true that $w' \models A_{k-1}(w)$ and $w' \not\models B_{k-1}(w)$. By fact 10 this implies $\sigma_{k-1}(w) =$

$\sigma_{k-1}(w')$ and by the induction hypothesis this implies $t_{k-1}(w) = t_{k-1}(w') \in \tau_2(t_k(m))$. Which proves $\tau_2(t_k(l)) \subseteq \tau_1(t_k(m))$.

Combining $\tau_1(t_k(m)) \subseteq \tau_1(t_k(l))$ and $\tau_2(t_k(l)) \subseteq \tau_2(t_k(m))$, one thus obtains $t_k(m) \leq t_k(l)$.

(3 $\Rightarrow$ 1) Assuming $t_k(m) \leq t_k(l)$, we will prove by induction on the complexity of $A \in \mathcal{L}$ (**IH2**) that $m \models A \quad \Rightarrow \quad l \models A$.

In the case that A is atomic, one can apply the corollary 20. For the other cases:

If $m \models A \wedge B$, then $m \models A$ and $m \models B$. Using the induction hypothesis (**IH2**) this implies $l \models A$ and $l \models B$ and hence $l \models A \wedge B$. Similarly $m \models A \vee B$ implies $l \models A \vee B$.

If $m \models A \rightarrow B$, then both A and B are formulas in $\mathcal{L}_{k-1}$. Assume that $l \not\models A \rightarrow B$, then there would be a $w \leq l$ such that $w \models A$ and $w \not\models B$. Now $t_{k-1}(w) \in \tau_1(t_k(l))$ and $\tau_1(t_k(l)) \subseteq \tau_1(t_k(m))$, so $t_{k-1}(w) \in \tau_1(t_k(m))$. Which would imply there is a $w' \geq m$ such that $t_{k-1}(w) = t_{k-1}(w')$. By the (original) induction hypothesis $Th_{k-1}(w) = Th_{k-1}(w')$, from which one could conclude that $m \not\models A \rightarrow B$, clearly a contradiction. Which proves $l \models A \rightarrow B$.

If $m \models A \backslash B$, again A and B are formulas in $\mathcal{L}_{k-1}$. Now for some $w \leq m$ it will be true that $w \models A$ and $w \not\models B$. So $w \in \tau_2(t_k(m)) \subseteq \tau_2(t_k(l))$. Hence there also is a $w' \leq l$ such that $t_{k-1}(w) = t_{k-1}(w')$. Again by the (original) induction hypothesis, one may conclude that $Th_{k-1}(w) = Th_{k-1}(w')$ and $l \models A \backslash B$. $\dashv$

Corollary 22 (A De Jongh/Jankov/Fine theorem for biIPL)
If m and l are nodes in finite Kripke models then:

- $l \models A_k(m) \quad \Leftrightarrow \quad t_k(m) \leq t_k(l)$
- $m \not\models B_k(l) \quad \Leftrightarrow \quad t_k(m) \leq t_k(l)$

Proof. By definition $l \models A_k(m)$ implies $A_k(l) \vdash A_k(m)$. Which by definition yields $\sigma_k(m) \leq \sigma_k(l)$. By theorem 21 this implies $t_k(m) \leq t_k(l)$. For the other direction, use theorem 21 to conclude $\sigma_k(m) \leq \sigma_k(l)$ from $t_k(m) \leq t_k(l)$. By definition this implies $A_k(l) \vdash A_k(m)$ and hence $l \models A_k(m)$.

If $m \not\models B_k(l)$ then $B_k(l) \vdash B_k(m)$ which implies $\sigma_k(m) \leq \sigma_k(l)$ according to lemma 9. Again, by theorem 21 it follows that $t_k(m) \leq t_k(l)$. For the other direction, if $t_k(m) \leq t_k(l)$, use theorem 21 to conclude $\sigma_k(m) \leq \sigma_k(l)$, which by lemma 9 implies $B_k(l) \vdash B_k(m)$ and hence $m \not\models B_k(l)$. $\dashv$

Usually the De Jongh/Jankov/Fine theorem is stated in terms of bisimulation (or p-morphism) between generated submodels of the nodes m and l. In the case of fragments with restricted nesting (in the modal equivalent of the theorem, nesting of $\Box$, otherwise nesting of implication) one would expect *layered bisimulations* (sometimes called *bounded bisimulation*). Such a layered bisimulation can easily be defined for **biIPL** too and it is not difficult to prove that $t_k(l) = t_k(m)$ iff l and m are k-bisimular.

3.3 Construction of exact models in biIPL

Theorem 21 relates the k-situations in the model $M_{\mathcal{L}_k}$ constructed in the proof of lemma 13 to the semantic types in the Kripke models for $\mathcal{L}_l$.

The model $M_{\mathcal{L}_k}$ not only contains all maximal situations in $\mathcal{L}_k$, but also all semantic k-types. Hence we can use the computation of all semantic k-types to obtain the model $M_{\mathcal{L}_k}$, which is a special case of a so called *exact Kripke model.*

Definition 23 *Let M be a Kripke model and $\mathcal{F}$ a fragment (in* **biIPL***). M is called the* exact Kripke model *of $\mathcal{F}$, $Exm(\mathcal{F})$ if:*

M **is universal for** $\mathcal{F}$ *For all formulas A and B in $\mathcal{F}$:*
$A \vdash B \quad \Leftrightarrow \quad M \models A \rightarrow B$

M **is differentiated for** F *For each upward closed subset X in M, there is a formula A such that $[\![A]\!] = X$.*

Exact Kripke models (for the $[\wedge, \rightarrow]^n$ and $[\wedge, \rightarrow, \neg]^n$ fragments of **IPL**) where first introduced by N.G. de Bruijn in [2]. Exact models as ordered sets of semantic types for other fragments of **IPL** can be found in [10].

Computer programs based on exact models can be used as fast theorem provers and have been applied for example in the research of fragments of logics (like in [11] and [12]).

Theorem 24 *The model $M_{\mathcal{L}_k^n}$ constructed in the proof of lemma 13 is the exact Kripke model of $\mathcal{L}_k^n$, the fragment of* **biIPL** *with n atoms and nesting of implication and subtraction restricted to k.*

Proof. That $M_{\mathcal{L}_k}$ is a universal model for $\mathcal{L}_k$ is an immediate consequence of (the construction in the proof of) lemma 11.

In $M_{\mathcal{L}_k}$ each maximal situation $\langle A, B\rangle$ generates a cone for $[\![A]\!]$, i.e. $[\![A]\!] = \uparrow\langle A, B\rangle$. As obviously $\langle A', B'\rangle \models A$ then by lemma 11 $A' \vdash A$ and hence $\langle A, B\rangle \leq \langle A', B'\rangle$.

On the other hand, each upward closed subset X of $M_{\mathcal{L}_k}$ is generated by a finite number of maximal situation $\langle A_i, B_i\rangle$ that are minimal in X in the ordering of $M_{\mathcal{L}_k}$. It easy to check that $X = [\![\bigvee A_i]\!]$. ⊣

The construction of $Exm(\mathcal{L}_0)$ is rather trivial. The 0-types are sets of atoms that can be ordered by the inclusion relation to obtain the exact model. Note that this is in fact the exact model of $[\wedge, \vee]^n$ as described in [10]. More precisely, we would have to take out the maximal element (the set of all atoms) as otherwise we would not have a formula for the empty set (as required according to the definition of exact Kripke model).

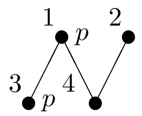
To construct $Exm(\mathcal{L}_{k+1})$ we use the model $Exm(\mathcal{L}_k)$.

- Recall that each $k + 1$-type t is of the form $\langle \tau_0(t), \tau_1(t), \tau_2(t)\rangle$, where $\tau_0(t)$ is a k-type and both $\tau_1(t)$ and $\tau_2(t)$ are sets of k-types. Moreover, $\tau_1(t) \subseteq \{t' \mid \tau_0(t) \leq t'\}$ in $Exm(\mathcal{L}_k)$ and, dually, $\tau_2(t) \subseteq \{t' \mid \tau_0(t) \geq t'\}$. This defines a finite search space for the construction of all $k + 1$-types.

- If $k > 1$ then for each $k+1$ type t we have a constraint (**TC**) on the choice of k-types in $\tau_1(t)$ and $\tau_2(t)$ as:
  $\tau_1(\tau_0(t)) = \{\tau_0(t') \mid t' \in \tau_1(t)\}$ and $\tau_2(\tau_0(t)) = \{\tau_0(t') \mid t' \in \tau_2(t)\}$.

One can easily turn the recipe of the above construction into a computer program constructing all semantic $k+1$-types and order these into a model $Exm(\mathcal{L}_{k+1})$.

Such a program computed the exact model of $\mathcal{L}^1_1$ for the fragment with one atomic formula as:

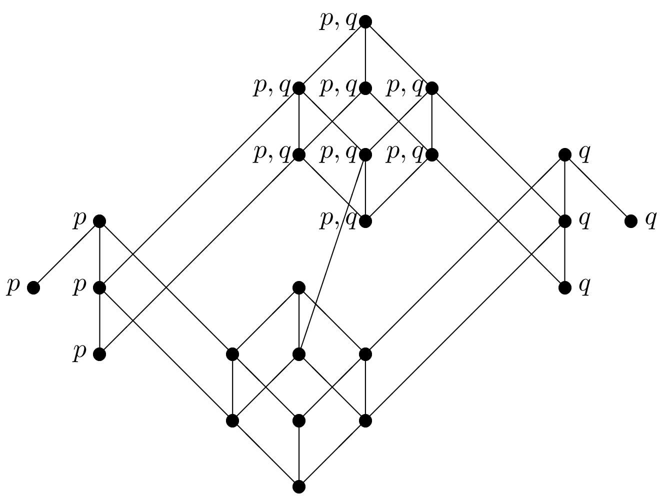


Using this exact model one can let the computer calculate all equivalence classes in $\mathcal{L}^1_1$, using the fact that the model is universal for this fragment. In the simple case of $\mathcal{L}_1$, one can still check by hand that the irreducible elements, the $A(m)$ of the 1-situations are:

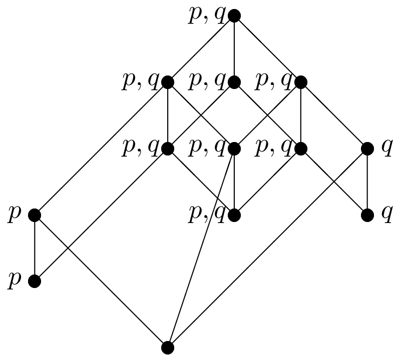
1	$p \wedge \neg\!\!\!- p$
2	$\neg p$
3	p
4	$\lnot\!\!\!- p$

Even the $Exm(\mathcal{L}^2_1)$ can still be constructed by hand.

Figure 1 *The exact model of* $\mathbf{biIPL}^2_1$.



Reducing the model above by disregarding the $\tau_1(t)$ part of the semantic types results in the dual of the exact model of $\mathbf{IPL}^2_1$ (see [10]), the exact model of $[\wedge, \vee, \setminus, \lnot\!\!\!-]^2_1$:



To prove that the types obtained by the recipe above are indeed all $k+1$-types, i.e. that for each such a constructed t there is a node m in a finite Kripke model such that $t = t_{k+1}(m)$, we sketch a procedure to construct such a node step by step.

Lemma 25 *Let m be a node in a finite Kripke model M. If $m \leq l$ and $t = \langle \tau_0(m), \tau_1(m) \setminus t_k(l), \tau_2(m) \rangle$ is a type construction for which the constraint* **TC**, *defined above, holds then there is a node m' in a finite Kripke model M' such that $t_{k+1}(m') = t$.*

Proof. For the proof we sketch the construction of M' from M. First take a copy of M where we remove $\uparrow l$. Let this new model be N. Next take the union of a disjunct copy of M and N and connect all maximal nodes in N that are 'new', i.e. do not correspond to maximal nodes in M with all successors of l in M excluding l itself.

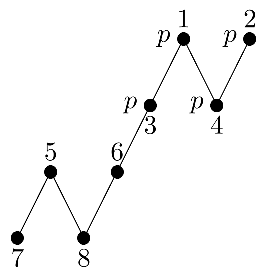
To check that the node m' in N corresponding to m in M has the $k+1$-type t can be proved by a simple induction on k. ⊣

Corollary 26 *Let m be a node in a finite Kripke model M. If $m \leq l$ and $t = \langle \tau_0(m), \tau_1(m), \tau_2(m) \setminus t_k(l) \rangle$ is a type construction for which the constraint* **TC**, *defined above, holds then there is a node m' in a finite Kripke model M' such that $t_{k+1}(m') = t$.*

Proof. Simply dualise the proof of lemma 25. ⊣

A computer program designed according to the construction as sketched above computed the exact models of $\mathcal{L}_2^1$ and $\mathcal{L}_3^1$ ($\mathbf{biIPL}_2^1$ and $\mathbf{biIPL}_3^1$).

Figure 2 *The exact model of $\mathcal{L}_2^1$.*



The corresponding formulas are:

1	$\lrcorner p \wedge \lrcorner\lrcorner p$	2	$\neg\lrcorner p$
3	$p \wedge \lrcorner p$	4	$\lrcorner\lrcorner p$
5	$\neg p \wedge \lrcorner\neg p$	6	$\neg\neg p \wedge \lrcorner p$
6	$\neg p$	7	$\lrcorner p \backslash \neg p$

The model $Exm(\mathcal{L}_3^1)$, with 32 nodes, is already too complicated for a simple drawing. Compared to the famous Rieger-Nishimura lattice that can be used as (an infinite) exact model for **IPL** (to be precise: after removing the minimal element for $p \to p$), this shows that even the fragment of **biIPL** with one atomic formula is much more complex than the corresponding fragment in **IPL**.

4 Fragments of biIPL

Fragments of **IPL** have been extensively studied, for example in [10]. Many results on the fragments of **IPL** can be extended to fragments of **biIPL** using the duality between $[\wedge, \to, \neg]$ and $[\vee, \backslash, \lrcorner]$.

Definition 27 *Define the following translation between formulas of* **biIPL***:*

$$
\begin{array}{ll}
\lceil p \rceil = p & \textit{if } p \textit{ atomic} \\
\lceil \top \rceil = \bot & \\
\lceil \bot \rceil = \top & \\
\lceil A \wedge B \rceil = \lceil A \rceil \vee \lceil B \rceil & \\
\lceil A \vee B \rceil = \lceil A \rceil \wedge \lceil B \rceil & \\
\lceil A \to B \rceil = \lceil B \rceil \backslash \lceil A \rceil & \\
\lceil A \backslash B \rceil = \lceil B \rceil \to \lceil A \rceil & \\
\lceil \Gamma \rceil = \{ \lceil C \rceil \mid C \in \Gamma \} &
\end{array}
$$

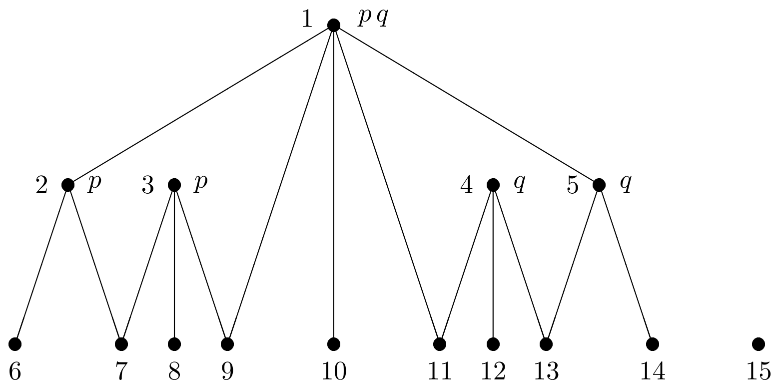
The following fact can easily be proved by checking that the steps in a proof of $A \vdash B$ can be replaced by their duals, to obtain a proof of $\lceil B \rceil \vdash \lceil A \rceil$ and vice versa.

Fact 28 *If A and B are formulas in* **biIPL** *then*

$$A \vdash B \quad \Leftrightarrow \quad \lceil B \rceil \vdash \lceil A \rceil$$

As a consequence, the Lindenbaum Algebra, i.e. the partial ordering of all equivalence classes, of a fragment is isomorphic to the Lindenbaum Algebra of its dual. For example, it is known that the **IPL** fragments $[\wedge, \rightarrow, \neg]^n$ and $[\vee, \wedge, \neg]^n$ are finite for any finite number of atoms n, i.e. have a finite Lindenbaum Algebra (see [10] or [16] for details).

Example 29 *The exact Kripke model of* $[\wedge, \rightarrow, \neg]^2$ *is:*



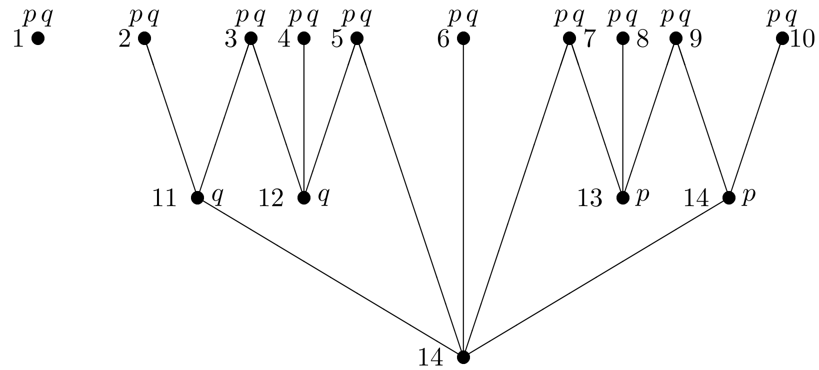
The corresponding formulas (the $A(m) \in [\wedge, \rightarrow, \neg]^2$) are:

1. $p \wedge q$
2. $p \wedge \neg\neg q$
3. $p \wedge \neg q$
4. $q \wedge \neg p$
5. $q \wedge \neg\neg p$
6. $\neg\neg q \wedge ((p \rightarrow q) \rightarrow p)$
7. $(\neg\neg q \rightarrow p) \wedge$ $((\neg\neg q \rightarrow q) \rightarrow p)$
8. $\neg(p \rightarrow q)$
9. $(q \rightarrow p) \wedge (\neg q \rightarrow p) \wedge (\neg\neg q \rightarrow q)$
10. $(p \leftrightarrow q) \wedge \neg\neg p$
11. $(p \rightarrow q) \wedge (\neg p \rightarrow q) \wedge (\neg\neg p \rightarrow p)$
12. $\neg(q \rightarrow p)$
13. $(\neg\neg p \rightarrow q) \wedge ((\neg\neg p \rightarrow p) \rightarrow q)$
14. $\neg\neg p \wedge ((q \rightarrow p) \rightarrow q)$
15. $\neg p \wedge \neg q$

Dually both $[\vee, \backslash, \lrcorner]^n$ and $[\wedge, \vee, \lrcorner]^n$ (and their subfragments) will be finite.

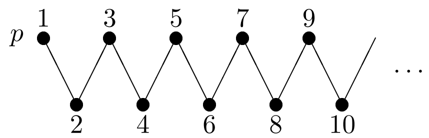
It turns out that if $\mathcal{F}$ is a fragment in **biIPL** that has an exact Kripke model $M_{\mathcal{F}}$, the exact Kripke of the dual fragment $\breve{F}$ can be constructed by changing the direction of the arrows and taking $atom(m)\breve{} = \{p \text{ atomic} \mid p \in \mathcal{F} \text{ and } p \notin atom(m)\}$.

Example 30 *The exact Kripke model of* $[\vee, \backslash, ⌐]^2$ *is:*



The fragments in **biIPL** that are dual to fragments in **IPL** with an infinite Lindenbaum Algebra are obviously infinite. So the fragments containing $[\wedge, \backslash]$ are infinite. The more interesting fragments are the *mixed* ones, like $[\neg, ⌐]$.

Example 31 *A model with infinitely many classes in* $[\neg, ⌐]$.



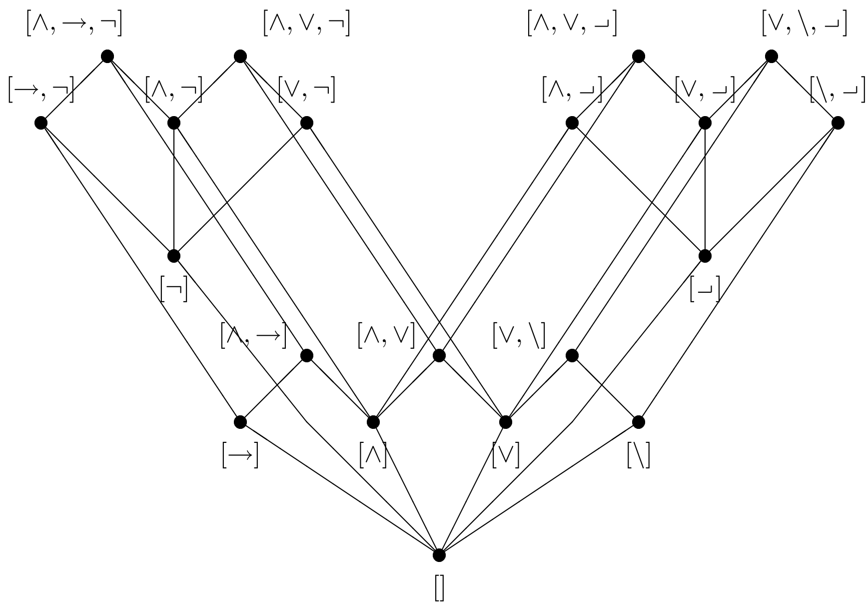
In example 31 one can easily check that:

$$
\begin{array}{lcl}
[\![p]\!] & = & \{1\} \\
[\![⌐\neg p]\!] & = & \{1, 2, 3\} \\
[\![⌐\neg⌐\neg p]\!] & = & \{1, 2, 3, 4, 5\}
\end{array}
$$

Or, more in general, if $A_1 = p$ and $A_{k+1} = ⌐\neg A_k$, then one can easily show that $[\![A_k]\!] = \{1, \ldots, 2 * k - 1\}$. Which proves that there are infinitely many different equivalence classes in $[\neg, ⌐]^1$.

Excluding all fragments that either contain $[\neg, ⌐]$, a subfragment of **IPL** which is known to be infinite, or the dual of such a fragment, the ordering of the remaining fragments in **biIPL** generated by a subset of the set of connectives $\{\wedge, \vee, \rightarrow, \neg, \backslash, ⌐\}$ is pictured in figure 3.

Figure 3 *The ordering of finite fragments in* **biIPL**.



As the maximal elements in this ordering, $[\wedge, \to, \neg]$, $[\wedge, \vee, \neg]$ and their duals $[\vee, \backslash, \lrcorner]$ and $[\wedge, \vee, \lrcorner]$ are fragments with a finite number of equivalence classes (once we fix a finite number of atomic formulas as generators), figure 3 shows all finite fragments in **biIPL** with connectives in the set $\{\wedge, \vee, \to, \neg, \backslash, \lrcorner\}$.

Apart from the fragments with connectives in the set of 'normal' connectives, $\{\wedge, \vee, \to, \neg, \backslash, \lrcorner\}$, one can define new connectives, like $\leftrightarrow$, $\neg\neg$ and their duals. As we know that for example the fragments $[\leftrightarrow, \neg]^n$ and $[\wedge, \to, \neg\neg]^n$ are finite, so will their duals.

The following tabel shows the finite fragments in the $[\wedge, \vee, \backslash, \lrcorner, \lrcorner\lrcorner]^n$ fragment of **biIPL** for $n \in \{1, 2, 3, 4\}$. These are the duals of the fragments in **IPL** studied in [10] and [16]).

fragment	$n=1$	$n=2$	$n=3$	$n=4$
$[\wedge]$	1	3	7	15
$[\vee]$	1	3	7	15
$[\wedge,\vee]$	1	4	18	166
$[\neg]$	3	6	9	12
$[\neg\neg]$	2	4	6	8
$[\wedge,\neg]$	7	385	$> 2^{70}$	
$[\vee,\neg]$	5	23	311	66 659
$[\wedge,\vee,\neg]$	7	626	$> 2^{70}$	
$[\wedge,\neg\neg]$	2	9	40	281
$[\vee,\neg\neg]$	2	8	26	80
$[\wedge,\vee,\neg\neg]$	2	19	1 889	
$[\to]$	2	14	25 165 802	$2^{623\,662\,965\,552\,393}$ $-50\,331\,618$
$[\wedge,\setminus]$	2	∞	∞	∞
$[\vee,\setminus]$	2	18	623 662 965 552 330	
$[\wedge,\vee,\setminus]$	2	∞	∞	∞
$[\setminus,\neg]$	6	518	$3 \times 2^{2\,148} - 546$	
$[\vee,\setminus,\neg]$	6	2 134	**D**	
$[\setminus,\neg\neg]$	4	252	$3 \times 2^{689} - 380$	
$[\wedge,\setminus,\neg\neg]$	5	∞	∞	∞
$[\vee,\setminus,\neg\neg]$	4	676	$> 2^{6\,383}$	
$[\wedge,\vee,\setminus,\neg\neg]$	5	∞	∞	∞

The number **D** is approximately $2^{6\,385}$ and the 1 923 digits of **D** have been calculated by a computer program of G.R. Renardel de Lavalette who dedicated this number to our mentor in research in propositional logics, Dick de Jongh.

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