The rules of intermediate logic

Rosalie Iemhoff *
Institute for Discrete Mathematics and Geometry E104, Technical University Vienna
Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria
iemhoff@logic.at
http://www.logic.at/people/iemhoff

For Dick de Jongh, on the occasion of his 65th birthday.

Abstract

If Visser's rules are admissible for an intermediate logic, they form a basis for the admissible rules of the logic. How to characterize the admissible rules of intermediate logics for which not all of Visser's rules are admissible is not known. Here we study the situation for specific intermediate logics. We provide natural examples of logics for which Visser's rule are derivable, admissible but non-derivable, or not admissible.

Keywords: Intermediate logics, admissible rules, realizability, Rieger-Nishimura formulas, Medvedev logic.

1 Introduction

Admissible rules, the rules under which a theory is closed, form one of the most intriguing aspects of intermediate logics. A rule $A/\ B$ is admissible for a theory if B is provable in it whenever A is. The rule $A/\ B$ is said to be derivable if the theory proves that $A \to B$. Classical propositional logic CPC does not have any non-derivable admissible rules, because in this case $A/\ B$ is admissible if and only if $A \to B$ is derivable, but for example intuitionistic propositional logic IPC has many admissible rules that

are not derivable in the theory itself. For example, the Independence of Premise rule *IPR*

$$\neg A \to B \vee C \ /\ (\neg A \to B) \vee (\neg A \to C)$$

is not derivable as an implication within the system, but it is an admissible rule of it. Therefore, knowing that $\neg A \to B \vee C$ is provable gives you much more than just that, because it then follows that also one of the stronger $(\neg A \to B)$ or $(\neg A \to C)$ is provable. Thus the admissible rules shed light on what it means

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to be constructively derivable, in a way that is not measured by the axioms or derivability in the theory itself.

We wonder what the situation is for logics in between classical and intuitionistic logic, the so-called intermediate logics. In particular, how do Visser's rules behave in other intermediate logics; are they admissible, derivable, do they form a basis? The special interest in Visser's rules stems from the fact that this collection of rules is the basis for the admissible rules of IPC (see below), as well as for some other well-known intermediate logics. In this paper we give an overview of the partial answers to the questions above and add some new observations.

This paper is dedicated to Dick, on the occasion of his 65th birthday. Dick has stirred my interest in intuitionistic logic, one of the subjects that I have loved ever since. I have learned a lot from him, not only through discussions on mathematics, but also in ways that are more difficult to describe. Dick possesses forces of teaching someone while being silent and only looking surprised, enthusiast or unconvinced. I am enjoying and benefiting from his remarks, whether silent or not, till today.

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2 Overview

Here we briefly summarize what is known about Visser's rules and intermediate logics. In the last section we provide the proofs of the observations below that are new. We will only be concerned with intermediate logics, i.e. logics between (possibly equal to) IPC and CPC.

The situation for IPC

First, let us briefly recall the situation for IPC. As said, this logic has many non derivable rules. In [8], using results from [5], it has been shown that the following rules form a basis for the admissible rules of IPC, i.e. that all admissible rules can be derived from Visser's rules and the theorems of IPC. Visser's rules is the collection of rules $V = \{V_n \mid \ldots n = 1, 2, 3, \ldots\}$, where

$$V_n \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \vee C \;/\; \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_j) \vee C.$$

The mentioned result is a syntactical characterization of the admissible rules of IPC. There are also result of a more computational nature: in [17] Rybakov showed that admissible derivability for IPC, $\mid\!\sim$, is decidable, and in the beautiful paper [6] Ghilardi presented a transparent algorithm for $\mid\!\sim$.

Remarks on Visser's rules

Visser's rules are an infinite collection of rules, that is, there is no n for which $V_{(n+1)}$ is derivable in IPC extended by the rule V_n [9]. Note that on the other hand V_n is derivable from $V_{(n+1)}$ for all n. In particular, if V_1 is not admissible for a logic, then none of Visser's rules are admissible. The independence of premise rule *IPR*

$$\neg A \to B \vee C \ / \ (\neg A \to B) \vee (\neg A \to C)$$

is a special instance of V_1. Having *IPR* admissible is strictly weaker than the admissibility of V_1; below we will see examples of logics for which the first one is admissible while the latter is not.

Note than when Visser's rules are admissible, then so are the rules

$$V_{nm} \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to \bigvee_{j=n+1}^{m} A_j) \vee C / \bigvee_{h=1}^{m} (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_h) \vee C.$$

As an example we will show that V_{13} is admissible for any logic for which V_1 is admissible. For simplicity of notation we take C empty. Assume that $\vdash_{\mathsf{L}} (A_1 \to B) \to A_2 \vee A_3 \vee A_4$. Then by V_1, reading $A_2 \vee A_3 \vee A_4$ as $A_2 \vee (A_3 \vee A_4)$,

$$\vdash_{\mathsf{L}} \big((A_1 \to B) \to A_1\big) \vee \big((A_1 \to B) \to A_2\big) \vee \big((A_1 \to B) \to A_3 \vee A_4\big).$$

A second application of V_1, with $C = \big((A_1 \to B) \to A_1\big) \vee \big((A_1 \to B) \to A_2\big)$, gives

$$\vdash_{\mathsf{L}} \bigvee_{i=1}^{2} \big((A_1 \to B) \to A_i\big) \vee \bigvee_{i=1,3,4} \big((A_1 \to B) \to A_i\big).$$

Therefore, $\vdash_{\mathsf{L}} \bigvee_{i=1}^{4} \big((A_1 \to B) \to A_i\big)$.

When Visser's rules are admissible

Somewhat surprisingly, at least to the author, it turns out that Visser's rules play an important role for other intermediate logics too.

Theorem 1 [10] If V is admissible for L then V is a basis for the admissible rules of L.

Thus, once Visser's rules are admissible we have a characterization of all admissible rules of the logic. Besides IPC, do there exist such logics? As it turns out, there indeed are. Even some well-known and natural ones (Section 4), e.g. the Gabbay-de Jongh logics $\mathsf{Bd_n}$, De Morgan logic KC, the Gödel logics $\mathsf{G_n}$, and Gödel-Dummett logic LC. For all these logics Visser's rules are admissible, and whence form a basis for their admissible rules.

Note that Theorem 1 in particular provides a condition for having no non-derivable admissible rules.

Corollary 2 If V is derivable for L then L has no non-derivable admissible rules.

The Gödel logics and Gödel-Dummett logic are in fact examples of this, as for these logics Visser's rules are not only admissible but also derivable. For the Gabbay-de Jongh logics and De Morgan logic one can show that this is not the case.

When are Visser's rules admissible?

Because of the theorem above, it would be useful to know when Visser's rules are admissible or not. At least for logics for which we have some knowledge about their models, a necessary condition for having Visser's rules admissible exist (for the definition of extension properties, see Section 3.3).

Theorem 3 [10] For any intermediate logic L, Visser's rules are admissible for L if and only if L has the offspring property.

Theorem 4 [10] For any intermediate logic L with the disjunction property, Visser's rules are admissible for L if and only if L has the weak extension property.

In fact, all the results on specific intermediate logics mentioned above, use these conditions for admissibility.

Disjunction property

A logic L has the *disjunction property* if

$$\vdash_{\mathsf{L}} A \vee B \ \Rightarrow \vdash_{\mathsf{L}} A \text{ or } \vdash_{\mathsf{L}} B.$$

The disjunction property plays an interesting role in the context of admissible rules. First of all, in combination with the admissibility of Visser's rules it characterizes IPC.

Theorem 5 [8] The only intermediate logic with the disjunction property for which all of Visser's rules are admissible is IPC.

This implies that if a logic has the disjunction property, not all of Visser's rules can be admissible. However, there is an instance of V_1 that will always be admissible in this case, namely *IPR*, see the section on Independence of Premise below.

For logics L that do have the disjunction property, $A \mathrel{|\!\sim}_{\mathsf{L}} C$ and $B \mathrel{|\!\sim}_{\mathsf{L}} C$ implies $A \vee B \mathrel{|\!\sim}_{\mathsf{L}} C$. Thus in the context of Visser's rules this e.g. implies that when the the following special instances of Visser's rules, the *restricted Visser rules*

$$V_n^- \quad (\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_{n+1} \vee A_{n+2}) \ / \ \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(A_i \to B_i) \to A_j),$$

are admissible for L, then so are Visser's rules. Therefore, when considering only logics with the disjunction property, like e.g. IPC, the difference between the Visser and the restricted Visser rules does not play a role. However, when considering intermediate logics in all generality, as we do in this paper, we cannot restrict ourselves to this sub-collection of Visser's rules.

When Visser's rule are not admissible

In the case that not all of Visser's rules are admissible we do not know of any general results concerning admissibility. We only have some partial results on specific intermediate logics, stating that some Visser rule is not admissible or that the logic in question has non-derivable admissible rules (Section 4). These results at least imply that

Fact 6 For every n, there are intermediate logics for which V_n is admissible while V_{n+1} is not, i.e. $V_1, \ldots, V_n$ are admissible and $V_{n+1}, V_{n+2} \ldots$ are not.

Fact 7 There are intermediate logics for which none of Visser's rules are admissible, but that do have non-derivable admissible rules.

The logics of (uniform) effective realizability UR and ER are examples of logics that have non-derivable admissible rules but for which V_1 is not admissible respectively derivable. Interestingly, for both these logics, the same special instance of V_1, namely the Independence of Premise rule *IPR* is a non-derivable admissible rule. That the rule is admissible in both logics is no coincedence, as the next section shows.

Independence of Premise

Although we have seen that there are not logics for which V_1 is not admissible, there is an instance of this rule that is admissible in any intermediate logic, namely *IPR*

$$\neg A \to B \vee C \ / \ (\neg A \to B) \vee (\neg A \to C).$$

Theorem 8 (Minari and Wronski [13]) For any intermediate logic L, we have ($\mathcal{H}$ is the class of Harrop formulas, see preliminaries):

$$\forall A \in \mathcal{H}: \ \mathsf{L} \vdash (A \to B \vee C) \ \Rightarrow \ \mathsf{L} \vdash (A \to B) \vee (A \to C).$$

Since any negation is a Harrop formula we have the following corollary.

Corollary 9 In any intermediate logic *IPR* is admissible.

Note that on the other hand we cannot conclude that for every Harrop formula A we have $(A \to B \vee C) \mathrel{|\!\sim}_{\mathsf{L}} (A \to B) \vee (A \to C)$, as the class of Harrop formulas is not closed under substitution.

General remarks

For completeness sake we include the following known facts about admissibility that states which rules might come up as admissible rules for a logic.

Fact 10 If $A \mathrel{|\!\sim}_{\mathsf{L}} B$, then $\mathsf{CPC} \vdash A \to B$.

Proof Suppose $A \mathrel{|\!\sim}_{\mathsf{L}} B$. This means that for all σ, $\vdash_{\mathsf{L}} \sigma A$ implies $\vdash_{\mathsf{L}} \sigma B$. Suppose the variables that occur in A and B are among $p_1 \dots p_n$. Consider $\sigma \in \{\top, \bot\}^n$. Note that for such σ, $\vdash_{\mathsf{CPC}} \sigma A$ iff $\vdash_{\mathsf{IPC}} \sigma A$ iff $\vdash_{\mathsf{L}} \sigma A$. Whence for all $\sigma \in \{\top, \bot\}^n$, if $\vdash_{\mathsf{CPC}} \sigma A$ then $\vdash_{\mathsf{CPC}} \sigma B$. Thus $\vdash_{\mathsf{CPC}} A \to B$. □

Corollary 11 If $A \mathrel{|\!\sim}_{\mathsf{L}} B$, then the logic that consists of L extended with the axiom scheme $(A \to B)$ is consistent.

Questions

There are too many questions to list them all, but among the most interesting general ones are the following three.

- If n is the largest n for which V_n is admissible for a logic with the disjunction property, do the rules $\{V_1, \dots, V_n\}$, i.e. $\{V_n\}$, form a basis for its admissible rules? And a similar question for the V_{mn} in case the logic does not have the disjunction property.
- Do there exist intermediate logics that have non-derivable admissible rules that are not instances of Visser's rules?
- Do there exist intermediate logics for which Visser's rules are admissible and the restricted Visser rules are not?

3 Preliminaries

Before we proceed with the proofs (in Section 4) of the new observations mentioned in the previous section, we have to settle some terminology and notation. As mentioned above, we will only be concerned with intermediate logics L, i.e. logics between (possibly equal to) IPC and CPC. We write $\vdash_{\mathsf{L}}$ for derivability in L. The letters A, B, C, D, E, F, H range over formulas, the letters p, q, r, s, t, range over propositional variables. We assume $\top$ and $\bot$ to be present in the language. $\neg A$ is defined as $(A \to \bot)$. We omit parentheses when possible; $\wedge$ binds stronger than $\vee$, which in turn binds stronger than $\to$. The class of *Harrop formulas* H is the class of formulas in which every disjunction occurs in the negative scope of an implication.

3.1 Admissible rules

A *substitution* σ in this paper will always be a map from propositional formulas to propositional formulas that commutes with the connectives. A *(propositional) admissible rule* of a logic L is a rule A/B such that adding the rule to the logic does not change the theorems of L, i.e.

$$\forall\sigma : \vdash_{\mathsf{L}} \sigma A \text{ implies } \vdash_{\mathsf{L}} \sigma B.$$

We write $A \mathrel{|\!\!\sim}_{\mathsf{L}} B$ if A/B is an admissible rule of L. The rule is called *derivable* if $A \vdash_{\mathsf{L}} B$ and *non-derivable* if $A \nvdash_{\mathsf{L}} B$. When R is the rule A/B, we write $R^{\rightarrow}$ for the implication $A \to B$. We say that a collection R of rules, e.g. V, is admissible for L if all rules in R are admissible for L. R is derivable for L if all rules in R are derivable for L. We write $A \vdash_{\mathsf{L}}^{R} B$ if B is derivable from A in the logic consisting of L extended with the rules R, i.e. there are $A = A_1, \ldots, A_n = B$ such that for all $i < n$, $A_i \vdash_{\mathsf{L}} A_{i+1}$ or there exists a σ such that $\sigma B_i/\sigma B_{i+1} = A_i/A_{i+1}$ and $B_i/B_{i+1} \in R$. If X and R are sets of admissible rules of L, then R is a *basis for* X if for every rule admissible rule $A \ / \ B$ in X we have $A \vdash_{\mathsf{L}}^{R} B$. If X consists of all the admissible rules of L, then R is called a *basis for the admissible rules of* L.

3.2 Kripke models

A Kripke models K is a triple $(W, \preccurlyeq, \Vdash)$, where W is a set (the set of *nodes*) with a unique least element that is called the *root*, $\preccurlyeq$ is a partial order on W and $\Vdash$, the *forcing relation*, a binary relation on W and sets of propositional variables. The pair $(W, \preccurlyeq)$ is called the *frame* of K. The notion of truth in a Kripke model is defined as usual. We write $K \models A$ if A is forced in all nodes of K and say that A *holds in* K. We write K_k for the model which domain consists of all nodes $k \preccurlyeq k'$ and which partial order and valuation are the restrictions of the corresponding relations of K to this domain.

3.3 Extension properties

For Kripke models $K_1, \ldots, K_n$, $(\sum_i K_i)'$ denotes the Kripke model which is the result of attaching one new node at which no propositional variables are forced, below all nodes in $K_1, \ldots, K_n$. $(\sum \cdot)'$ is called the *Smorynski operator*. Two models K, K' are *variants* of each other, written KvK', when they have the same set of nodes and partial order, and their forcing relations agree on all nodes except possibly the root. A class of models U has the *extension property* if for every finite family of models $K_1, \ldots, K_n \in U$, there is a variant of $(\sum_i K_i)'$ which belongs to U. U has the *weak extension property* if for every model $K \in U$, and every finite collection of nodes $k_1, \ldots, k_n \in K$ distinct from the root, there exists a model $M \in U$ such that

$$\exists M_1 \left((\sum_i K_{k_i})' v M_1 \ \wedge \ (M_1 \twoheadrightarrow M) \right).$$

U has the *offspring* property if for every model $K \in U$, and for every finite collection of nodes $k_1, \ldots, k_n \in K$ distinct from the root, there exists a model $M \in U$ such that

$$\exists M_1 \exists M_0 \big((\sum_i K_{k_i})' v M_1 \ \wedge \ (M_1 + K)' v M_0 \ \wedge \ (M_0 \twoheadrightarrow M) \big).$$

A logic L has the extension (weak extension, offspring) property if it is sound and complete with respect to some class of models that has the extension (weak extension, offspring) property. Note that for all three properties the class of models involved does not have to be the class of *all* models of L. However, we might as well require that, because in [10] it has been shown that if a logic has the offspring property, then so does the class of all its models. Since the class of all models of a logic is closed under submodels and bounded morphic images, this also implies that for logics

$$\text{extension property} \ \Rightarrow \ \text{offspring property} \ \Rightarrow \ \text{weak extension property.}$$

The reason that we have chosen the definition of offspring property as given above, not the most elegant one, is that it will turn out particularly useful for the application to various frame complete logics discussed in the last section. There are quite natural classes of models that satisfy the offspring property, e.g. the class of linear models, as the reader may wish to verify for himself.

If we would not restrict our models to rooted ones, the extension property and the weak extension property would be equivalent, at least for logics. Since we require our Kripke models to be rooted, there is a subtle difference between the two:

Fact 12 If a logic L has the extension property, it has the disjunction property.

As there are logics that do not have the disjunction property, but that have the weak extension property, the latter is indeed stronger. We will see examples of such logics in Section 4.

4 Results

In this section we collect the results on specific intermediate logics discussed in the introduction. We present proofs of the observations that are new, and refer to the literature for the ones that have been obtained before. Below follows the list of intermediate logics involved. As the reader can see, it consists mainly of quite well-known and natural logics, whatever the word natural might exactly mean. This is not accidently so, as we are particularly interested in these kind of logics. For it might well be that for specific purposes, e.g. for showing that there exist logics for which *IPR* is admissible while V_1 is not, one can cook up a logic that serves as an example, but we feel that to come up with a well-known and natural instance of such a logic is somehow much more satisfying.

A point of terminology: when we say "axiomatized by ..." we mean "axiomatized over IPC by ...". For a class of frames F, L is called the *logic of the frames F* when L is sound and complete with respect to F.

The principle $IPR^{\to}$ is denoted *IP* and called Independence of Premise:

$$IP \qquad (\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C).$$

$\mathsf{Bd_n}$ The logic of frames of depth at most n. $\mathsf{Bd_1}$ is axiomatized by $bd_1 = A_1 \vee \neg A_1$, and $\mathsf{Bd_{n+1}}$ by $bd_{n+1} = (A_{n+1} \vee (A_{n+1} \to bd_n))$ [2].

$\mathsf{D_n}$ The Gabbay-de Jongh logics [4], axiomatized by the following scheme: $\bigwedge_{i=0}^{n+1}((A_i \to \bigvee_{j \neq i} A_j) \to \bigvee_{j \neq i} A_j) \to \bigvee_{i=0}^{n+1} A_i$. $\mathsf{D_n}$ is complete with respect to the class of finite trees in which every point has at most $(n+1)$ immediate successors.

$\mathsf{G_k}$ The Gödel logics, first introduced in [7]. They are extensions of LC axiomatized by $A_1 \vee (A_1 \to A_2) \vee \ldots \vee (A_1 \wedge \ldots \wedge A_{k-1} \to A_k)$. $\mathsf{G_k}$ is the logic of the linearly ordered Kripke frames with at most $k-1$ nodes [1].

KC De Morgan logic (also called Jankov logic), axiomatized by $\neg A \vee \neg\neg A$. The logic of the frames with one maximal node.

KP The logic axiomatized by *IP*, i.e. by $(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C)$. The logic is called Kreisel-Putnam logic. It constituted the first counterexample to Łukasiewicz conjecture that IPC is the greatest intermediate logic with the disjunction property [11].

LC Gödel-Dummett logic [3], the logic of the linear frames. It is axiomatized by the scheme $(A \to B) \vee (B \to A)$.

ML Medvedev logic [12]. The logic of the frames $F_1, F_2, \ldots$, where the nodes of F_n are the nonempty subsets of $\{1, \ldots, n\}$ and $\preccurlyeq$ is $\supseteq$.

$\mathsf{M_n}$ The logic of frames with at most n maximal nodes. Note that $\mathsf{M_1} = \mathsf{KC}$.

$\mathsf{NL_n}$ The logics axiomatized by formulas in one propositional variable (so-called Nishimura formulas nf_n). $\mathsf{NL_n}$ is axiomatized by nf_n, where $nf_0 = \bot$, $nf_1 = p$, $nf_2 = \neg p$, $nf_{2n+1} = nf_{2n} \vee nf_{2n-1}$, and $nf_{2n+2} = nf_{2n} \to nf_{2n-1}$.

ER The logic of effectively realizable formulas: the logic consisting of formulas $A(p_1, \ldots, p_n)$ for which there exists a recursive function f such that for any substitution of the p_i by arithmetical formulas φ_i with Gödel numbers m_i, $f(m_1, \ldots, m_n)$ realizes the result, i.e. $\mathbb{N} \models$"$f(m_1, \ldots, m_n)\mathtt{r}A(\varphi_1, \ldots, \varphi_n)$". There is no r.e. axiomatization known for this logic, but it is known that it is a proper extension of IPC [16].

UR The logic of formulas that are effectively realizable by a constant function, i.e. the logic consisting of formulas $A(p_1, \ldots, p_n)$ such that there exists a number e such that for any substitution of the p_i by arithmetical formulas φ_i, e realizes the result, i.e. $\mathbb{N} \models$"$e\mathbf{r}A(\varphi_1, \ldots, \varphi_n)$". There is no r.e. axiomatization known for this logic, but it was shown in [16] that it is a proper extension of IPC.

Sm The greatest intermediate logic properly included in classical logic. It is axiomatized by $\big((A \to B) \vee (B \to A)\big) \wedge (A \vee (A \to B \vee \neg B))$ and it is complete with respect to frames of at most 2 nodes [2].

TF The logic of frames with at most three nodes.

V The logic axiomatized by $V_1^{\to}$, or equivalently, axiomatized by the implication corresponding to the rule V_1^-: $\big((A_1 \to B) \to A_2 \vee A_3\big) \to \bigvee_{i=1}^{3} \big((A_1 \to B) \to A_i\big)$.

Theorem 13 [10] Visser's rules are derivable in Bd_1, $\mathsf{G_k}$, LC, Sm and V. Hence these logics do not have non-derivable admissible rules.

Theorem 14 Visser's rules form a basis for the admissible rules of the logics KC, $\mathsf{M_n}$ and TF. Visser's rules are not derivable in any of these logics.

Proof For the first two logics the statement has been proved in [10]. For TF we consider the class of frames of the logic: F_1 consists of one node, F_2 of two nodes $k_0 \preccurlyeq k_1$ and $F_3 = (\{k_0, k_1, k_2\}, \{(k_0, k_1), (k_{0,2}\}$. Therefore, pick a model K based on one of these frames. We treat the case that the frame is F_3 and leave the other cases to the reader. Pick nodes $l_1, \ldots, l_n$ in K distinct from the root. We have to show that there is a variant M_1 of $(\Sigma_i K_{l_i})'$ such that a bounded morphic image M of a variant M_0 of $(M_1 + K)'$ has at most three nodes. If $n = 1$, say $l_1 = k_1$, we force at the root m_i of the variant M_i the same atoms as at k_i. When we let M be the restriction of K to domain k_0, k_1,then M is a bounded morhic image of M_0. Namely, define the bounded morhism f via $f(m_i) = k_i$, where f sends K and K_{k_1} to the corresponding parts of M. If $n = 2$, we can force at the roots m_1, m_0 of the variants M_1, M_0 the same atoms as at k_0. We leave it to the reader to verify that K is a bounded morphic image of M_0.

To see that Visser's rules are not derivable in TF, we leave it to the reader to construct appropriate countermodels to $IPR^{\to}$, i.e.

$$(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C),$$

which is an instance of $V_1^{\to}$. Whence none of Visser's rules can be derivable, because $V_n^{\to}$ clearly implies $V_1^{\to}$. □

Note that all the logics in the previous theorem are examples of logics which have the weak extension property, but not the 2-extension property, as they do not have the disjunction property (see Fact 12). That they do not have the disjunction property follows from the fact that the only logic with the disjunction property for which all Visser's rules are admissible is IPC.

Next we consider intermediate logics for which a full characterization of their admissible rules is not known. First we give some examples of logics for which not all of Visser's rules are admissible but that have non-derivable admissible rules: the logics $\mathsf{D_n}$ ($n \geq 1$) and UR. For the logic ML we only know that it has non-derivable admissible rules and that none of Visser's rules are derivable. But whether Visser's rule are admissible we do not know. Finally, there follow some examples of logics for which the Visser rules or the restricted Visser rules are not admissible, but for which we do not know whether they have non-derivable admissible rules at all: KP and NL_n ($n \geq 9$).

First an obsevation that we will often use below. Allthough the statement can also be derived directly from Theorem 5, the proof as given here contains a funny self-application of V_1, which is the reason we have included the proof here.

Proposition 15 If an intermediate logic L has the disjunction property, V_1 is not derivable in L. Hence none of Visser's rules are derivable in L.

Proof Suppose L has the disjunction property and that V_1 is derivable in L. Thus for $X = (p_1 \to q)$, L derives the following instance of V_1,

$$\mathsf{L} \vdash (X \to p_2 \vee p_3) \to \bigvee_{i=1}^{3} (X \to p_i).$$

Since V_1 is derivable, it is certainly admissible. Thus so is V_{13} (see the Remarks on Visser's rules in the Introduction). Applying the rule (now with $A_1 = (X \to p_2 \vee p_3)$ and $A_i = (X \to p_i)$ for $i > 1$) then gives

$$\mathsf{L} \vdash \big((X \to p_2 \vee p_3) \to X\big) \vee \bigvee_{i=1}^{3} \big((X \to p_2 \vee p_3) \to (X \to p_i)\big).$$

Since L has the disjunction property, this would imply that at least one of $\big((X \to p_2 \vee p_3) \to (X \to p_i)\big)$, or $\big((X \to p_2 \vee p_3) \to X\big)$ is derivable in L. However, these formulas are not even derivable in classical logic. $\Box$

Theorem 16 [10] The restricted Visser rules are admissible but not derivable for $\mathsf{Bd_n}$ for $n \geq 2$.

Theorem 17 [8] For the logics $\mathsf{D_n}$ ($n \geq 1$), V_{n+1} is admissible, while V_{n+2} is not. In none of the logics V_1 is derivable.

Proposition 18 (with Jaap van Oosten) V_1 is not admissible for UR. *IPR* is a non-derivable admissible rule of UR and ER (and thus V_1 is not derivable in ER).

Proof It is convenient to assume that our coding of pairs and recursive functions is such that $\langle 0, 0\rangle = 0$ and $0 \cdot x = 0$ for all x($a \cdot b$ denotes the result of applying the a-th partial recursive function to b). Then 0 realizes every negation of a sentence that has no realizers.

First, we show that V_1 is not admissible for UR. In [16] G.F. Rose showed that the following formula, not derivable in IPC, belongs to UR: for $A = \neg p \vee \neg q$,

$$\mathsf{UR} \vdash \big((\neg\neg A \to A) \to \neg\neg A \vee \neg A\big) \to \neg\neg A \vee \neg A.$$

Let $B = \big((\neg\neg A \to A) \to \neg\neg A \vee \neg A\big)$. If the 1st Visser rule, V_1, would be admissible, this would imply that

$$\mathsf{UR} \vdash (B \to \neg\neg A) \vee (B \to \neg A) \vee (B \to (\neg\neg A \to A)).$$

The fact that UR has the disjunction property, plus some elementary logic, leads to

$$\mathsf{UR} \vdash (B \to \neg\neg A) \text{ or } \mathsf{UR} \vdash (B \to \neg A) \text{ or } \mathsf{UR} \vdash (\neg\neg A \to A).$$

As classical logic does not even derive $(B \to \neg\neg A)$ or $(B \to \neg A)$, certainly $\mathsf{UR} \nvdash (B \to \neg\neg A)$ and $\mathsf{UR} \nvdash (B \to \neg A)$. Also $\mathsf{UR} \nvdash (\neg\neg A \to A)$. For if not, there is a realizer e of every substitution instance $\neg\neg(\neg\varphi \vee \neg\psi) \to \neg\varphi \vee \neg\psi$ of $(\neg\neg A \to A)$. From this we derive a contradiction as follows. Thus for all x such that $x\mathbf{r}\neg\neg(\neg\varphi \vee \neg\psi)$, $(e \cdot x)_0 = 0$ and $(e \cdot x)_1\mathbf{r}\neg\varphi$, or $(e \cdot x)_0 = 1$ and $(e \cdot x)_1\mathbf{r}\neg\psi$. Take $\varphi = \bot$ and $\psi = \top$. Let $\chi = (\neg\varphi \vee \neg\psi)$ and $\chi' = (\neg\psi \vee \neg\varphi)$. Note that $\forall y\neg(y\mathbf{r}\neg\chi)$ and $\forall y\neg(y\mathbf{r}\neg\chi')$. Since for all ϕ

$$x\mathbf{r}\neg\neg\phi \;\leftrightarrow \forall y\neg(y\mathbf{r}\neg\phi),$$

this implies that every number, in particular 0, is a realizer of $\neg\neg\chi$ and $\neg\neg\chi'$. Whence $(e \cdot 0)$ is a realizer of both χ and χ'. If $(e \cdot 0)_0 = 0$, then $(e \cdot 0)_1\mathbf{r}\neg\psi$, and if $(e \cdot 0)_0 = 1$, then $(e \cdot 0)_1\mathbf{r}\neg\psi$ too. As $\neg\psi$ cannot have a realizer, we have reached the desired contradiction.

To show that $\neg A \to B_0 \vee B_1 \mathrel{|\!\sim}_{\mathsf{UR}} (\neg A \to B_0) \vee (\neg A \to B_1)$, assume that $\mathsf{UR} \vdash \neg A \to B_0 \vee B_1$, for some A, B_0, B_1, and suppose that the atoms that occur in A, B_0, B_1 are $p_1, \ldots, p_n$. So there is a number e such that for all $\psi_1, \ldots, \psi_n$, e realizes $(\neg A \to B_0 \vee B_1)(\psi_1, \ldots, \psi_n)$. We write $A(\bar\psi)$ for $A(\psi_1, \ldots, \psi_n)$, and similarly for B_0, B_1. We have to construct a realizer that, for all $\psi_1, \ldots, \psi_n$, realizes

$$(\neg A(\bar\psi) \to B_0(\bar\psi)) \vee (\neg A(\bar\psi) \to B_1(\bar\psi)). \tag{1}$$

Since we reason classically, as we consider uniform effective realizability, either $\exists x(x\mathbf{r}\neg A(\bar\psi))$ or $\forall x\neg(x\mathbf{r}\neg A(\bar\psi))$. Thus by the definition of realizability, $\forall x\neg(x\mathbf{r}A(\bar\psi))$ or $\forall x\neg(x\mathbf{r}\neg A(\bar\psi))$. In the first case, $e \cdot 0\mathbf{r}(B_0(\bar\psi) \vee B_1(\bar\psi))$. Thus for $i = 0, 1$, if $(e \cdot 0)_0 = i$, $(e \cdot 0)_1\mathbf{r}B_i((\bar\psi)$, whence if d is the code of the program that always outputs $(e \cdot 0)_1$, then $< d, i >$ realizes (1). In the second case, $\forall x\neg(x\mathbf{r}\neg A(\bar\psi))$, $< e, 0 >$ realizes (1), as $\neg A(\bar\psi)$ has no realizers.

Jaap van Oosten in [15] showed that *IPR* is not derivable in ER, which implies that *IPR* is a non-derivable admissible rule of both ER and UR by Corollary 9. Thus, to finish the proof of the theorem, it remains to prove the non-derivability of *IPR* in ER. We repeat van Oosten's proof, as given in [15]:

Let $A(f)$ be the sentence $\forall x \exists y T(f,x,y)$ and let $B(f)$ and $C(f)$ be negative sentences, expressing "there is an x on which f is undefined, and the least such x is even" (respectively, odd). Suppose there is a total recursive function F such that for every f, $F(f)$ realizes

$$(\neg A(f) \to B(f) \vee C(f)) \ \to \ ((\neg A(f) \to B(f)) \vee (\neg A(f) \to C(f))).$$

Choose, by the recursion theorem, an index f of a partial recursive function of two variables, such that:
$f{\cdot}(g,x) = 0$ if there is no $w \leq x$ witnessing that $F(S^1_1(f,g)){\cdot}g$ is defined, or if x is the least such witness, and *either* $(F(S^1_1(f,g)){\cdot}g)_0 = 0$ and x is even, *or* $(F(S^1_1(f,g)){\cdot}g)_0 \neq 0$ *and* x is odd;
$f{\cdot}(g,x)$ is undefined in all other cases.
Then for every g we have:

- $F(S^1_1(f,g)){\cdot}g$ is defined. For otherwise, $f{\cdot}(g,x) = 0$ for all x, hence $S^1_1(f,g)$ is total, so g realizes

  $$\neg A(S^1_1(f,g)) \to B(S^1_1(f,g)) \vee C(S^1_1(f,g)),$$

  which would imply that $F(S^1_1(f,g)){\cdot}g$ is defined, a contradiction;
- If $(F(S^1_1(f,g)){\cdot}g)_0 = 0$ then the first number on which $S^1_1(f,g)$ is undefined is odd, so $C(S^1_1(f,g))$ holds;
- If $(F(S^1_1(f,g)){\cdot}g)_0 \neq 0$ then $B(S^1_1(f,g))$ holds.

Now let, again by the recursion theorem, g be chosen such that for all y:

$$g{\cdot}y = \begin{cases} \langle 1,0\rangle & \text{if } (F(S^1_1(f,g)){\cdot}g)_0 = 0 \\ \langle 0,0\rangle & \text{if } (F(S^1_1(f,g)){\cdot}g)_0 \neq 0 \end{cases}$$

Then g is a realizer for $\neg A(S^1_1(f,g)) \to [B(S^1_1(f,g)) \vee C(S^1_1(f,g))]$. However, it is easy to see that $F(S^1_1(f,g)){\cdot}g$ makes the wrong choice
This finishes van Oosten's proof that *IPR* is not derivable in ER, and thereby the proposition is proved. □

Proposition 19 V_1 is not derivable in ML. *IPR* is derivable in ML.

Proof That V_1 is not derivable in ML follows from Proposition 15, because the logic has the disjunction property. To see that *IPR* is derivable in L, i.e. that *IP* is a principle of ML, we use the frame characterization of ML given above. The proof is left to the reader. □

As mentioned above, we do not know whether Visser's rules are admissible in ML. For the following logics we do not know whether they have non-derivable admissible rules, although we know that Visser's rules are not admissible.

Proposition 20 [10] V_1 is not admissible for KP.

Proposition 21 For the logics $\mathsf{NL_n}$, V_1 is not admissible for $n \geq 9$. For the even $n \geq 9$ the restricted rule V_1^- is not admissible too. Visser's rules are non-derivable and form a basis for $n = 5, 8$. Visser's rules are derivable for $n \leq 4$ and $n = 6$. We do not know what happens for $n = 7$.

Proof Observe that for $n = 0, 1, 2, 4$ the logic is inconsistent ($nf_4 \equiv \neg\neg p$), for $n = 5, 8$ it is equal to KC [14], and for $n = 3, 6$ it is CPC ($nf_6 \equiv \neg\neg p \to p$, substituting $A \vee \neg A$ for p shows that the corresponding logic is CPC). This treats the cases $n \leq 6$ and $n = 8$. For $n \geq 9$ we show that V_1 is not admissible for $\mathsf{NL_n}$. Since for even $n \geq 10$ the logics $\mathsf{NL_n}$ have the disjunction property [18], this will imply that V_1^- is not admissible for $n \geq 10$ (see the section on the disjunction property), and whence prove the theorem.
To prove that V_1 is not admissible, we will use the following fact.

Fact 22 [14] $\mathsf{NL_n} \nvdash \mathsf{NL_m}$ for all $7 \leq m < n$.
For all l, for all $k \geq l + 3$: $\mathsf{IPC} \vdash (nf_l \to nf_k)$.
For all l: $\mathsf{IPC} \vdash (nf_{2l+2} \vee nf_{2l} \equiv nf_{2l+3})$.

The main ingredient of the proof is the following claim.
Claim For all n, if V_1 is admissible for NL_n, then for all even $k \geq 8$, for all A,

$$\mathsf{NL}_n \vdash nf_k \vee A \;\Rightarrow\; \mathsf{NL}_n \vdash nf_{k-4} \vee nf_{k-6} \vee A. \tag{2}$$

Proof of the Claim Assume V_1 is admissible for NL_n and $\mathsf{NL}_n \vdash nf_k$ for some even $k \geq 8$. Note that the assumption that $k \geq 8$ guarantees that $nf_{k-8}, \ldots, nf_k$ are all well-defined. Since k is even

$$nf_k = nf_{k-2} \to nf_{k-3} = (nf_{k-4} \to nf_{k-5}) \to nf_{k-4} \vee nf_{k-5}.$$

Thus we can apply V_1 to $nf_k \vee A$ and obtain

$$\mathsf{NL}_n \vdash (nf_{k-2} \to nf_{k-4}) \vee (nf_{k-2} \to nf_{k-5}) \vee A. \tag{3}$$

Using that

$$nf_{k-2} \to nf_{k-5} = (nf_{k-4} \to nf_{k-5}) \to nf_{k-6} \vee nf_{k-7},$$

we can apply V_1 to (3) again. This gives

$$\mathsf{NL}_n \vdash (nf_{k-2} \to nf_{k-4}) \vee (nf_{k-2} \to nf_{k-6}) \vee (nf_{k-2} \to nf_{k-7}) \vee A.$$

Consider the first disjunct $(nf_{k-2} \to nf_{k-4})$. Since $nf_{k-4} = nf_{k-6} \to nf_{k-7}$, this disjunct is equivalent to $nf_{k-2} \wedge nf_{k-6} \to nf_{k-7}$. Using Fact 22, it follows that this is equivalent to $nf_{k-6} \to nf_{k-7}$. Again by Fact 22, $\mathsf{IPC} \vdash nf_{k-6} \to nf_{k-2}$. Therefore, the third disjunct implies $nf_{k-6} \to nf_{k-7}$. All this gives

$$\mathsf{NL}_n \vdash (nf_{k-6} \to nf_{k-7}) \vee (nf_{k-2} \to nf_{k-6}) \vee (nf_{k-6} \to nf_{k-7}) \vee A.$$

Using the definition of the nf's this gives

$$\mathsf{NL}_n \vdash nf_{k-4} \vee (nf_{k-2} \to nf_{k-6}) \vee A. \tag{4}$$

Finally, we have to distinguish two cases. If $k \geq 9$, similar considerations as above show that the second disjunct of (4) is equivalent to $nf_{k-8} \to nf_{k-9} = nf_{n-6}$. This leads to

$$\mathsf{NL}_n \vdash nf_{k-4} \vee nf_{k-6} \vee A. \tag{5}$$

If $k = 8$, the second disjunct of (4) is $nf_6 \to nf_2 = nf_6 \wedge p \to \bot$, which is equivalent to $\neg p = nf_{k-6}$, as $(p \to nf_{k-6})$ by Fact 22. This also leads to (5), as desired. This proves (2), and thereby the claim. □

We continue with the proof of the theorem by showing that for all $n \geq 9$, the assumption that V_1 is admissible for NL_n leads to a contradiction. We treat the odd and even cases separately.

First, assume V_1 is admissible for NL_n, for some even $n \geq 10$. Since $\mathsf{NL}_n \vdash nf_n$, application of the Claim (take A empty) gives

$$\mathsf{NL}_n \vdash nf_{n-4} \vee nf_{n-6}. \tag{6}$$

We distinguish the cases $n = 10$ and $n \geq 12$. If $n = 10$, we have

$$nf_{n-4} \vee nf_{n-6} = nf_6 \vee nf_4 \equiv nf_7.$$

The equivalence follows from Fact 22. Together with (6) this implies $\mathsf{NL}_{10} \vdash \mathsf{NL}_7$, contradicting Fact 22. For the case of the even $n \geq 12$, a second application of the Claim, with $A = nf_{n-6}$, to (6) leads to $\mathsf{NL}_n \vdash nf_{n-8} \vee nf_{n-10} \vee nf_{n-6}$. Note that we can apply the Claim because $n \geq 12$ implies that $n - 4 \geq 8$. By Fact 22,

$$\mathsf{IPC} \vdash (nf_{n-6} \vee nf_{n-8} \vee nf_{n-10}) \to (nf_{n-3} \vee nf_{n-2} \vee nf_{n-1}).$$

As $(nf_{n-3} \vee nf_{n-2} \vee nf_{n-1}) \equiv nf_{n-1}$, we can conclude $\mathsf{NL}_n \vdash nf_{n-1}$, and thus $\mathsf{NL}_n \vdash \mathsf{NL}_{n-1}$, which contradicts Fact 22.

Second, assume V_1 is admissible for NL_n, for some odd $n \geq 9$. Observe that $\mathsf{NL}_n \vdash nf_{n-1} \vee nf_{n-2}$. Applying the Claim (with $A = nf_{n-2}$) gives

$$\mathsf{NL}_n \vdash nf_{n-5} \vee nf_{n-7} \vee nf_{n-2}. \tag{7}$$

Since $nf_{n-2} = nf_{n-3} \vee nf_{n-4}$ and $nf_{n-4} = nf_{n-5} \vee nf_{n-6}$ this gives

$$\mathsf{NL}_n \vdash nf_{n-3} \vee nf_{n-5} \vee nf_{n-6} \vee nf_{n-7}. \tag{8}$$

If $n = 9$, this disjunction is equal to $nf_6 \vee (nf_4 \vee nf_3) \vee nf_2 \equiv nf_6 \vee nf_5 \vee nf_2$. Using Fact 22 this is again equivalent to $nf_6 \vee nf_5 = nf_7$. Thus (8) gives $\mathsf{NL}_9 \vdash \mathsf{NL}_7$, contradicting Fact 22. For the odd $n \geq 11$, we apply the Claim again to (8), with $A = nf_{n-5} \vee nf_{n-6} \vee nf_{n-7}$. This can be done as $n \geq 11$, whence $n - 3 \geq 8$. This leads to

$$\mathsf{NL}_n \vdash nf_{n-5} \vee nf_{n-6} \vee nf_{n-7} \vee nf_{n-9}.$$

By Fact 22 this implies $\mathsf{NL}_n \vdash nf_{n-1}$, and thus $\mathsf{NL}_n \vdash \mathsf{NL}_{n-1}$, which contradicts Fact 22. □

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