Dualities for Some Intuitionistic Modal Logics

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Abstract

We present a duality for the intuitionistic modal logic **IK** introduced by Fischer Servi in [8, 9]. Unlike other dualities for **IK** reported in the literature (see for example [13]), the dual structures of the duality presented here are ordered topological spaces endowed with just *one* extra relation, which is used to define the set-theoretic representation of both $\Box$ and $\Diamond$. Also, this duality naturally extends the definitions and techniques used by Fischer Servi in the proof of completeness for **IK** via canonical model construction [10]. We also give a parallel presentation of dualities for the intuitionistic modal logics $\mathbf{IntK}_\Box$ and $\mathbf{IntK}_\Diamond$. Finally, we turn to the intuitionistic modal logic **MIPC**, which is an axiomatic extension of **IK**, and we give a very natural characterization of the dual spaces for **MIPC** introduced in [2] as a subcategory of the category of the dual spaces for **IK** introduced here.

1 Preliminaries

1.1 The logics $\mathbf{IntK}_\Box$, $\mathbf{IntK}_\Diamond$ and IK

Let **Int** be the standard intuitionistic propositional calculus. For a non-empty set M of unary modal operators, let $\mathcal{L}_M$ be the intuitionistic propositional language augmented by the connectives in M. By an *intuitionistic modal logic* we understand any subset of $\mathcal{L}_M$ containing all the theorems of **Int** and closed under modus ponens, substitution and the regularity rule $\phi \to \psi / m\phi \to m\psi$ for every $m \in M$.

The logic $\mathbf{IntK}_\Box$, in the language $\mathcal{L}_\Box$, is axiomatized by adding the following axioms to **Int**:

$$\Box(\phi \wedge \psi) = \Box\phi \wedge \Box\psi \quad \text{and} \quad \Box\top = \top.$$

The logic $\mathbf{IntK}_\Diamond$, in the language $\mathcal{L}_\Diamond$, is axiomatized by adding the following axioms to **Int**:

$$\Diamond(\phi \vee \psi) = \Diamond\phi \vee \Diamond\psi \quad \text{and} \quad \Diamond\bot = \bot.$$

The logic $\mathbf{IntK}_{\Box\Diamond}$ is the smallest logic $\mathcal{S}$ in the language $\mathcal{L}_{\Box\Diamond}$ such that $\mathbf{IntK}_\Box \cup \mathbf{IntK}_\Diamond \subseteq \mathcal{S}$. The modal operators $\Box$ and $\Diamond$ are independent in $\mathbf{IntK}_{\Box\Diamond}$, but are connected in the logic **IK**, defined by Fischer Servi in [8, 9] and axiomatized in [10]. **IK** is the axiomatic extension of $\mathbf{IntK}_{\Box\Diamond}$ obtained by adding the following *connecting axioms*:

$$\Diamond(\phi \to \psi) \to (\Box\phi \to \Diamond\psi) \text{ and } (\Diamond\phi \to \Box\psi) \to \Box(\phi \to \psi).$$

1.2 Algebraic semantics

Definition 1.2.1. (IntK$_\Box$-algebra) $\mathcal{A} = \langle A, \wedge, \vee, \to, \Box, 0, 1\rangle$ *is an* **IntK$_\Box$**-algebra *iff* $\langle A, \wedge, \vee, \to, 0, 1\rangle$ *is a Heyting algebra and the following axioms are satisfied:*

$$\Box(a \wedge b) = \Box a \wedge \Box b \quad \textit{and} \quad \Box 1 = 1.$$

Definition 1.2.2. (IntK$_\Diamond$-algebra) $\mathcal{A} = \langle A, \wedge, \vee, \to, \Diamond, 0, 1\rangle$ *is an* **IntK$_\Diamond$**-algebra *iff* $\langle A, \wedge, \vee, \to, 0, 1\rangle$ *is a Heyting algebra and the following axioms are satisfied:*

$$\Diamond(a \vee b) = \Diamond a \vee \Diamond b \quad \textit{and} \quad \Diamond 0 = 0.$$

Definition 1.2.3. (IK-algebra) $\mathcal{A} = \langle A, \wedge, \vee, \to, \Box, \Diamond, 0, 1\rangle$ *is an* **IK**-algebra *iff* $\langle A, \wedge, \vee, \to, 0, 1\rangle$ *is a Heyting algebra and the following axioms are satisfied:*

1. $\Box 1 = 1$	*2.* $\Diamond 0 = 0$
3. $\Box(a \wedge b) = \Box a \wedge \Box b$	*4.* $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$
5. $\Diamond(a \to b) \leq \Box a \to \Diamond b$	*6.* $\Diamond a \to \Box b \leq \Box(a \to b)$.

2 Frames

An *intuitionistic frame* [4] is a poset, i.e. a structure $\langle X, \leq\rangle$, such that $X \neq \emptyset$ and $\leq$ is a reflexive, antisymmetric and transitive binary relation on X. Let $\mathcal{P}_\leq(X)$ be the collection of the $\leq$-increasing subsets of X. For every relation $S \subseteq X \times X$ and every $Y, Z \subseteq X$, let

$$\begin{aligned}
\Box_S(Y) &= \{x \in X \mid S[x] \subseteq Y\} \\
\Diamond_S(Y) &= \{x \in X \mid S[x] \cap Y \neq \emptyset\} \\
Z \Rightarrow_S Y &= \Box_S((X \setminus Z) \cup Y) \\
&= \{x \in X \mid \forall y \in X(xSy \;\&\; y \in Z \Rightarrow y \in Y\}.
\end{aligned}$$

Lemma 2.0.4. *For every poset* $\langle X, \leq\rangle$ *and every* $A, B \in \mathcal{P}_\leq(X)$, $A \Rightarrow_\leq B \in \mathcal{P}_\leq(X)$.

Proof. Assume that $x \in (A \Rightarrow_\leq B)$ and $x \leq y$. Then for every $z \in A$, if $y \leq z$, then $x \leq z$, and so $z \in B$. This shows that $y \in (A \Rightarrow_\leq B)$. □

Lemma 2.0.5. *For every intuitionistic frame* $\langle X, \leq\rangle$, $\langle \mathcal{P}_\leq(X), \cap, \cup, \Rightarrow_\leq, \emptyset, X\rangle$ *is a Heyting algebra.*

Proof. For every partial order $\langle X, \leq\rangle$, it holds that $\langle \mathcal{P}_\leq(X), \cap, \cup, \emptyset, X\rangle$ is a bounded distributive lattice. Let us show that for every $A, B, C \in \mathcal{P}_\leq(X)$,

$$(A \cap C) \subseteq B \text{ iff } C \subseteq (A \Rightarrow_\leq B).$$

($\Rightarrow$) Let $c \in C$, and let us show that $c \in A \Rightarrow_{\leq} B$, i.e. that if $c \leq y$ and $y \in A$, then $y \in B$. As $c \leq y$, $c \in C$ and C is $\leq$-increasing, then $y \in C$, so $y \in A \cap C \subseteq B$.
($\Leftarrow$) If $x \in A \cap C \subseteq C \subseteq A \Rightarrow_{\leq} B$, then for every $y \in A$ such that $x \leq y$, $y \in B$. Then take $y = x$. □

Definition 2.0.6. (Frames) *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be such that X is a nonempty set, $\leq$ is a preorder on X and R is a binary relation.*

1. *$\mathcal{F}$ is an* $\mathbf{IntK}_{\Box}$-frame *iff* $(\leq \circ R) \subseteq (R \circ \leq)$.
2. *$\mathcal{F}$ is an* $\mathbf{IntK}_{\Diamond}$-frame *iff* $(\geq \circ R) \subseteq (R \circ \geq)$.
3. *$\mathcal{F}$ is an* $\mathbf{IK}$-frame *iff* $(\geq \circ R) \subseteq (R \circ \geq)$ *and* $(R \circ \leq) \subseteq (\leq \circ R)$.

Example 2.0.7. *For every partial order $\langle X, \leq\rangle$,*

1. *$\langle X, \leq, \leq\rangle$ is an $\mathbf{IntK}_{\Box}$-frame.*
2. *$\langle X, \leq, \geq\rangle$ is an $\mathbf{IntK}_{\Diamond}$-frame.*
3. *$\langle X, \leq, \geq \circ \leq\rangle$ is an $\mathbf{IK}$-frame.*

Lemma 2.0.8. *For every partial order $\langle X, \leq\rangle$ and every binary relation S on X,*

1. *the following are equivalent:*
   (a) $(\leq \circ S) \subseteq (S \circ \leq)$.
   (b) *$\mathcal{P}_{\leq}(X)$ is closed under $\Box_S$.*
2. *The following are equivalent:*
   (a) $(\geq \circ S) \subseteq (S \circ \geq)$.
   (b) *$\mathcal{P}_{\leq}(X)$ is closed under $\Diamond_S$.*
3. *The following are equivalent:*
   (a) $(S \circ \leq) \subseteq (\leq \circ S)$.
   (b) *For every $x \in X$, $S[x\uparrow] \in \mathcal{P}_{\leq}(X)$.*

Proof. 1. (a $\Rightarrow$ b) Let us show that if $Y \subseteq X$ is $\leq$-increasing, $S[x] \subseteq Y$ and $x \leq y$, then $S[y] \subseteq Y$: For every $z \in S[y]$, $x \leq ySz$, hence by assumption $v \leq z$ for some $v \in S[x] \subseteq Y$, and as Y is $\leq$-increasing, $z \in Y$.
(b $\Rightarrow$ a) Assume that $x \leq ySz$, and let us show that $z \in S[x]\uparrow$. As $S[x]\uparrow$ is $\leq$-increasing, then by assumption $\Box_S(S[x]\uparrow) = \{s \in X \mid S[s] \subseteq S[x]\uparrow\}$ is $\leq$-increasing. As $S[x] \subseteq S[x]\uparrow$, then $x \in \Box_S(S[x]\uparrow)$, hence $x \leq y$ implies that $y \in \Box_S(S[x]\uparrow)$, and as $z \in S[y] \subseteq S[x]\uparrow$, then $z \in S[x]\uparrow$.
2. (a $\Rightarrow$ b) Let us show that if $Y \subseteq X$ is $\leq$-increasing, $S[x] \cap Y \neq \emptyset$ and $x \leq y$, then $S[y] \cap Y \neq \emptyset$: let $z \in S[x] \cap Y$, then $y \geq xSz$, hence by assumption $v \geq z$ for some $v \in S[y]$, and as $z \in Y$ and Y is $\leq$-increasing, then $v \in Y$.

(b $\Rightarrow$ a) Assume that $x \geq ySz$, and let us show that $z \in S[x]{\downarrow}$. As $S[x]{\downarrow}$ is $\leq$-decreasing, then $S[x]{\downarrow}^c$ is $\leq$-increasing, so by assumption $\Diamond_S(S[x]{\downarrow}^c) = \{s \in X \mid S[s] \nsubseteq S[x]{\downarrow}\}$ is $\leq$-increasing. As $S[x] \subseteq S[x]{\downarrow}$, then $x \notin \Diamond_S(S[x]{\downarrow}^c)$, hence $x \geq y$ implies that $y \notin \Diamond_S(S[x]{\downarrow}^c)$, and as $z \in S[y] \subseteq S[x]{\downarrow}$, then $z \in S[x]{\downarrow}$.

3. (a $\Leftarrow$ b) Let us show that if $z \in S[x{\uparrow}]$ and $z \leq y$, then $y \in S[x{\uparrow}]$: As $x \leq vSz \leq y$ for some $v \in X$, then by assumption $x \leq v \leq wSy$, hence $y \in S[x{\uparrow}]$.

(b $\Rightarrow$ a) Assume that $xSy \leq z$, and let us show that $z \in S[x{\uparrow}]$. As $y \in S[x] \subseteq S[x{\uparrow}]$, $y \leq z$ and $S[x{\uparrow}]$ is $\leq$-increasing by assumption, then $z \in S[x{\uparrow}]$. $\square$

Corollary 2.0.9. *For every preorder $\langle X, \leq\rangle$ and every binary relation R on X, $\mathcal{P}_{\leq}(X)$ is closed under $\Box_{(\leq \circ R)}$.*

Proof. It holds that $(\leq \circ (\leq \circ R)) \subseteq ((\leq \circ R) \circ \leq)$, hence clause (a) of item 1 of 2.0.8 is satisfied with $S = (\leq \circ R)$. $\square$

Lemma 2.0.10. *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be a relational structure.*

1. *If $\mathcal{F}$ is an $\mathbf{IntK}_\Box$-frame, then $\mathcal{A}_\mathcal{F} = \langle \mathcal{P}_{\leq}(X), \cap, \cup, \Rightarrow, \Box_R, \emptyset, X\rangle$ is an $\mathbf{IntK}_\Box$-algebra. Hence, every subalgebra $\mathcal{A}$ of $\mathcal{A}_\mathcal{F}$ is an $\mathbf{IntK}_\Box$-algebra.*

2. *If $\mathcal{F}$ is an $\mathbf{IntK}_\Diamond$-frame, then $\mathcal{A}_\mathcal{F} = \langle \mathcal{P}_{\leq}(X), \cap, \cup, \Rightarrow, \Diamond_R, \emptyset, X\rangle$ is an $\mathbf{IntK}_\Diamond$-algebra. Hence, every subalgebra $\mathcal{A}$ of $\mathcal{A}_\mathcal{F}$ is an $\mathbf{IntK}_\Diamond$-algebra.*

3. *If $\mathcal{F}$ is an $\mathbf{IK}$-frame, then $\mathcal{A}_\mathcal{F} = \langle \mathcal{P}_{\leq}(X), \cap, \cup, \Rightarrow, \Box_{(\leq\circ R)}, \Diamond_R, \emptyset, X\rangle$ is an $\mathbf{IK}$-algebra. Hence, every subalgebra $\mathcal{A}$ of $\mathcal{A}_\mathcal{F}$ is an $\mathbf{IK}$-algebra.*

Proof. 3. Let us show that $\Diamond_R(U \Rightarrow V) \subseteq (\Box_{(\leq\circ R)}U \Rightarrow \Diamond_R V)$ for every $U, V \in \mathcal{P}_{\leq}(X)$: Assume that $x \in \Diamond_R(U \Rightarrow V)$, let $x \leq z$ and $z \in \Box_{(\leq\circ R)}U$, and let us show that $z \in \Diamond_R V$, i.e. that $R[z] \cap V \neq \emptyset$. As $x \in \Diamond_R(U \Rightarrow V)$, then there exists $y \in R[x] \cap (U \Rightarrow V)$, hence $z \geq xRy$, and so, as $\mathcal{F}$ is an $\mathbf{IK}$-frame, $zRv \geq y$ for some $v \in X$. As $v \in R[z] \subseteq (\leq \circ R)[z] \subseteq U$, $y \leq v$ and $y \in (U \Rightarrow V)$, then $v \in V$, and as $v \in R[z]$, then $R[z] \cap V \neq \emptyset$.

Let us show that $(\Diamond_R U \Rightarrow \Box_{(\leq\circ R)}V) \subseteq \Box_{(\leq\circ R)}(U \Rightarrow V)$ for every $U, V \in \mathcal{P}_{\leq}(X)$: Assume that $x \in (\Diamond_R U \Rightarrow \Box_{(\leq\circ R)}V)$, let $z \in (\leq\circ R)[x]$ and $z \leq y \in U$, and let us show that $y \in V$. As $z \in (\leq \circ R)[x]$, then $x \leq vRz \leq y$, hence, as $\mathcal{F}$ is an $\mathbf{IK}$-frame, $x \leq v \leq wRy$ for some $w \in X$. As $wRy \in U$, then $w \in \Diamond_R U$, and as $x \leq w$, then $w \in \Box_{(\leq\circ R)}V$, hence $y \in R[w] \subseteq (\leq \circ R)[w] \subseteq Y$. $\square$

3 Topological semantics

Definition 3.0.11. (General frame) *A* general frame *is a structure $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ such that X is a nonempty set, $\leq$ is a partial order on X, R is a binary relation on X, and $\mathcal{A}$ is a subalgebra of $\langle \mathcal{P}_{\leq}(X), \cap, \cup, \Rightarrow, \emptyset, X\rangle$. For every general frame $\mathcal{G}$, $\mathcal{F}_\mathcal{G} = \langle X, \leq, R\rangle$ is the associated frame, and the associated ordered topological space $\mathbf{X}_\mathcal{G} = \langle X, \leq, \tau_\mathcal{A}\rangle$ has the following subbase: $\{Y \mid Y \in \mathcal{A}\} \cup \{(X \setminus Y) \mid Y \in A\}$.*

3.1 General $\mathbf{IntK}_\Box$-frames and their morphisms

Definition 3.1.1. (General $\mathbf{IntK}_\Box$-frame) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is a* general $\mathbf{IntK}_\Box$-frame *iff*

D1. $\mathbf{X}_\mathcal{G}$ *is an Esakia space, and $\mathcal{A}$ is the collection of the clopen increasing sets of* $\mathbf{X}_\mathcal{G}$.

D2'. *$\mathcal{A}$ is closed under* $\Box_R$.

D3. *For every $x \in X$,* $R[x] \in K(\mathbf{X}_\mathcal{G})$.

Definition 3.1.2. (p-morphism of general $\mathbf{IntK}_\Box$-frames) *Let $\mathcal{G}_i = \langle X_i, \leq_i, R_i, \mathcal{A}_i\rangle$ be general $\mathbf{IntK}_\Box$-frames, $i = 1, 2$. A map $f : X_1 \to X_2$ is a* p-morphism *iff for every $x, x', y \in X_1$, $z \in X_2$,*

M1. *if $x \leq_1 y$ then $f(x) \leq_2 f(y)$.*

M2. *If $f(x) \leq_2 z$ then $f(x') = z$ for some $x' \in x\uparrow$.*

M3. *For every $Y \in \mathcal{A}_2$, $f^{-1}[Y] \in \mathcal{A}_1$.*

M4. *If xR_1y then $f(x)R_2f(y)$.*

M5'. *If $f(x)R_2z$ then $f(x') \leq_2 z$ for some $x' \in R_1[x]$.*

Conditions M1–M3 together are equivalent to saying that $f : \mathbf{X}_{\mathcal{G}_1} \longrightarrow \mathbf{X}_{\mathcal{G}_2}$ is a continuous and strongly isotone map.

3.2 General $\mathbf{IntK}_\Diamond$-frames and their morphisms

Definition 3.2.1. (General $\mathbf{IntK}_\Diamond$-frame) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is a* general $\mathbf{IntK}_\Diamond$-frame *iff*

D1. $\mathbf{X}_\mathcal{G}$ *is an Esakia space, and $\mathcal{A}$ is the collection of the clopen increasing sets of* $\mathbf{X}_\mathcal{G}$.

D2. *$\mathcal{A}$ is closed under* $\Diamond_R$.

D3. *For every $x \in X$,* $R[x] \in K(\mathbf{X}_\mathcal{G})$.

Definition 3.2.2. (p-morphism of general $\mathbf{IntK}_\Diamond$-frames) *Let $\mathcal{G}_i = \langle X_i, \leq_i, R_i, \mathcal{A}_i\rangle$ be general $\mathbf{IntK}_\Diamond$-frames, $i = 1, 2$. A map $f : X_1 \to X_2$ is a* p-morphism *iff for every $x, x', y \in X_1$, $z \in X_2$,*

M1. *if $x \leq_1 y$ then $f(x) \leq_2 f(y)$.*

M2. *If $f(x) \leq_2 z$ then $f(x') = z$ for some $x' \in x\uparrow$.*

M3. *For every $Y \in \mathcal{A}_2$, $f^{-1}[Y] \in \mathcal{A}_1$.*

M4. *If xR_1y then $f(x)R_2f(y)$.*

M5. *If $f(x)R_2z$ then $z \leq_2 f(x')$ for some $x' \in R_1[x]$.*

3.3 General IK-frames and their morphisms

Definition 3.3.1. (General IK-frame) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is a* general **IK**-frame *iff*

D1. *$\mathbf{X}_\mathcal{G}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen increasing sets of $\mathbf{X}_\mathcal{G}$.*

D2. *$\mathcal{A}$ is closed under $\Diamond_R$ and $\Box_{(\leq \circ R)}$.*

D3. *For every $x \in X$, $R[x] \in K(\mathbf{X}_\mathcal{G})$.*

D4. *For every $x \in X$, $R[x\uparrow] \in K^\uparrow(\mathbf{X}_\mathcal{G})$.*

Example 3.3.2. *For every finite partial order $\langle X, \leq\rangle$, the general frame $\mathcal{G} = \langle X, \leq, (\geq \circ \leq), \mathcal{P}_\leq(X)\rangle$ is a general* **IK***-frame.*

Proof. Let τ be the topology generated by taking $\mathcal{P}_\leq(X) \cup \mathcal{P}_\geq(X)$ as a subbase. As X is finite, then $\mathbf{X} = \langle X, \leq, \tau\rangle$ is compact. For every $U \in \mathcal{P}_\leq(X)$, U is clopen and $\leq$-increasing. Viceversa, if U is clopen and $\leq$-increasing, then $U \in \mathcal{P}_\leq(X)$, so $\mathcal{P}_\leq(X)$ is the collection of the clopen increasing subsets of $\mathbf{X}$. $\mathbf{X}$ is totally order-disconnected, for if $x \not\leq y$, then $y \notin x\uparrow \in \mathcal{P}_\leq(X)$, so $\mathbf{X}$ is a Priestley space[1]. $\mathbf{X}$ is an Esakia space, for if U is a clopen subset of $\mathbf{X}$, then $U\downarrow \in \mathcal{P}_\geq(X)$, hence $U\downarrow$ is clopen. Item 2 of 2.0.8 implies that $\mathcal{P}_\leq(X)$ is closed under $\Diamond_{(\geq \circ \leq)}$, and by 2.0.9, $\mathcal{P}_\leq(X)$ is closed under $\Box_{\leq \circ (\geq \circ \leq)}$. For every $x \in X$, $(\geq \circ \leq)[x] = x\downarrow\uparrow \in \mathcal{P}_\leq(X)$ and $(\geq \circ \leq)[x\uparrow] = x\uparrow\downarrow\uparrow \in \mathcal{P}_\leq(X)$, so they are clopen increasing, therefore $(\geq \circ \leq)[x] \in K(\mathbf{X})$ and $(\geq \circ \leq)[x\uparrow] \in K^\uparrow(\mathbf{X})$. □

Definition 3.3.3. (p-morphism of general IK-frames) *Let $\mathcal{G}_i = \langle X_i, \leq_i, R_i, \mathcal{A}_i\rangle$ be general* **IK***-frames, $i = 1, 2$. A map $f : X_1 \to X_2$ is a* p-morphism *iff for every $x, x', y \in X_1$, $z \in X_2$,*

M1. *if $x \leq_1 y$ then $f(x) \leq_2 f(y)$.*

M2. *If $f(x) \leq_2 z$ then $f(x') = z$ for some $x' \in x\uparrow$.*

M3. *For every $Y \in \mathcal{A}_2$, $f^{-1}[Y] \in \mathcal{A}_1$.*

M4. *If xR_1y then $f(x)R_2f(y)$.*

M5. *If $f(x)R_2z$ then $z \leq_2 f(x')$ for some $x' \in R_1[x]$.*

M6. *If $f(x)(\leq_2 \circ R_2)z$ then $f(x') \leq_2 z$ for some $x' \in R_1[x\uparrow]$.*

[1] Actually, τ is the discrete topology, because, as X is finite, then every closed set is the finite intersection of clopen sets and so every closed set is clopen. Moreover, as $\mathbf{X}$ is a Priestley space, then it is Hausdorff, so every singleton set is closed and therefore clopen, so every subset of X is clopen, for it is the finite union of clopen sets.

4 From general L-frames to algebras

For every general frame $\mathcal{G} = \langle X, \leq, R, \mathcal{A} \rangle$, let $\mathcal{G}^+ := \mathcal{A}$, and for every continuous map $f : \mathbf{X}_{\mathcal{G}_1} \longrightarrow \mathbf{X}_{\mathcal{G}_2}$ let $f^+ : \mathcal{G}_2^+ \longrightarrow \mathcal{G}_1^+$ be given by the assignment $Y \longmapsto f^{-1}[Y]$ for every $Y \in \mathcal{A}_{\mathcal{G}_2}$.

4.1 The action of $(\_)^+$ on objects

Let us recall that for every general frame $\mathcal{G} = \langle X, \leq, R, \mathcal{A} \rangle$, $\mathcal{F}_{\mathcal{G}} = \langle X, \leq, R \rangle$ is the associated frame.

Lemma 4.1.1. *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A} \rangle$ be a general frame.*

1. *If $\mathcal{G}$ is a general* $\mathbf{IntK}_\Box$*-frame, then $\mathcal{F}_{\mathcal{G}}$ is an* $\mathbf{IntK}_\Box$*-frame.*
2. *If $\mathcal{G}$ is a general* $\mathbf{IntK}_\Diamond$*-frame, then $\mathcal{F}_{\mathcal{G}}$ is an* $\mathbf{IntK}_\Diamond$*-frame.*
3. *If $\mathcal{G}$ is a general* **IK***-frame, then $\mathcal{F}_{\mathcal{G}}$ is an* **IK***-frame.*

Proof. 1. Let us show that for every $x \in X$, $(\leq \circ R)[x] \subseteq (R \circ \leq)[x]$: Suppose that $z \in (\leq \circ R)[x]$ and $z \notin (R \circ \leq)[x] = R[x]{\uparrow}$ for some $z \in X$. As $z \notin R[x]{\uparrow}$, then $y \not\leq z$ for every $y \in R[x]$, hence, by D1, for every $y \in R[x]$ there exists a clopen increasing subset U_y of $\mathbf{X}_{\mathcal{G}}$ such that $y \in U_y$ and $z \notin U_y$, and so $R[x] \subseteq \bigcup_{y \in R[x]} U_y$, and as $\mathbf{X}_{\mathcal{G}}$ is compact and $R[x]$ is closed by D3, then $R[x] \subseteq \bigcup_{i=1}^n U_{y_i} = U$ for some $y_1, \ldots, y_n \in R[x]$. As U is clopen increasing, then $U \in \mathcal{A}$, moreover, $z \notin U$ and $R[x] \subseteq U$.
As $z \in (\leq \circ R)[x]$, then $x \leq wRz$ for some $w \in X$. Since $z \in (R[w] \setminus U)$, then $w \notin \Box_R U \in \mathcal{A}$ by D2', so in particular $\Box_R U$ is increasing, and as $x \leq w$, then $x \notin \Box_R U$, i.e. $R[x] \not\subseteq U$, contradiction.
2. Let us show that for every $x \in X$, $(\geq \circ R)[x] \subseteq (R \circ \geq)[x]$: Suppose that $z \in (\geq \circ R)[x]$ and $z \notin (R \circ \geq)[x] = R[x]{\downarrow}$ for some $z \in X$. As $z \notin R[x]{\downarrow}$, then $z \not\leq y$ for every $y \in R[x]$, hence, by D1, for every $y \in R[x]$ there exists a clopen decreasing subset V_y of $\mathbf{X}_{\mathcal{G}}$ such that $y \in V_y$ and $z \notin V_y$, and so $R[x] \subseteq \bigcup_{y \in R[x]} V_y$, and as $\mathbf{X}_{\mathcal{G}}$ is compact and $R[x]$ is closed by D3, then $R[x] \subseteq \bigcup_{i=1}^n V_{y_i} = V$ for some $y_1, \ldots, y_n \in R[x]$. Let $U = (X \setminus V)$. As U is clopen increasing, then $U \in \mathcal{A}$, moreover, $z \in U$ and $R[x] \cap U = \emptyset$.
As $z \in (\geq \circ R)[x]$, then $x \geq wRz$ for some $w \in X$. Since $z \in R[w] \cap U$, then $w \in \Diamond_R U \in \mathcal{A}$ by D2, so in particular $\Diamond_R U$ is increasing, and as $w \leq x$, then $x \in \Diamond_R U$, i.e. $R[x] \cap U \neq \emptyset$, contradiction.
3. Let us show that for every $x \in X$, $(R \circ \leq)[x] \subseteq (\leq \circ R)[x]$: Suppose that $z \in (R \circ \leq)[x]$ and $z \notin (\leq \circ R)[x] = R[x{\uparrow}]$ for some $z \in X$. As $z \notin R[x{\uparrow}]$ which is a closed and increasing subset of $\mathbf{X}_{\mathcal{G}}$ by D4, then $y \not\leq z$ for every $y \in R[x{\uparrow}]$, hence, by D1, for every $y \in R[x{\uparrow}]$ there exists a clopen increasing subset U_y of $\mathbf{X}_{\mathcal{G}}$ such that $y \in U_y$ and $z \notin U_y$, and so $R[x{\uparrow}] \subseteq \bigcup_{y \in R[x]} U_y$, and as $\mathbf{X}_{\mathcal{G}}$ is compact and $R[x]$ is closed by D3, then $R[x] \subseteq \bigcup_{i=1}^n U_{y_i} = U$ for some $y_1, \ldots, y_n \in R[x]$. As U is clopen increasing, then $U \in \mathcal{A}$, moreover, $z \notin U$ and $R[x{\uparrow}] \subseteq U$.

As $z \in (R \circ \leq)[x]$, then $xRw \leq z$ for some $w \in X$. Since $w \in R[x] \subseteq R[x\uparrow] \subseteq U$, then $w \in U$ which is increasing, and as $w \leq z$, then $z \in U$, contradiction. □

Proposition 4.1.2. *Let* $\mathbf{L} \in \{\mathbf{IntK}_\Box, \mathbf{IntK}_\Diamond, \mathbf{IK}\}$. *For every general* $\mathbf{L}$*-frame* $\mathcal{G} = \langle X, \leq, R, \mathcal{A} \rangle$, $\mathcal{A}$ *is an* $\mathbf{L}$*-algebra.*

Proof. It immediately follows from 2.0.10 and 4.1.1. □

4.2 The action of $(\_)^+$ on arrows

Proposition 4.2.1. *Let* $\mathbf{L} \in \{\mathbf{IntK}_\Box, \mathbf{IntK}_\Diamond, \mathbf{IK}\}$. *For every p-morphism* $h : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ *of general* $\mathbf{L}$*-frames,* $h^+ : {\mathcal{G}_2}^+ \longrightarrow {\mathcal{G}_1}^+$ *is a homomorphism of* $\mathbf{L}$*-algebras.*

Proof. If $h : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is a p-morphism of general $\mathbf{L}$-frames, then in particular it is a continuous and strongly isotone map between the Esakia spaces $\mathbf{X}_{\mathcal{G}_1}$ and $\mathbf{X}_{\mathcal{G}_2}$, hence from the duality for Heyting algebras, h^+ is a homomorphism between the Heyting algebra reducts of ${\mathcal{G}_2}^+$ and ${\mathcal{G}_1}^+$. Let us show that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are general $\mathbf{IntK}_\Box$-frames, then for every $Y \in \mathcal{A}_2$,

$$h^{-1}[\Box_{R_2} Y] = \Box_{R_1} h^{-1}[Y].$$

For every $x \in X_1$, $x \in h^{-1}[\Box_{R_2} Y]$ iff $R_2[h(x)] \subseteq Y$, and $x \in \Box_{R_1} h^{-1}[Y]$ iff $R_1[x] \subseteq h^{-1}[Y]$.

($\subseteq$) Assume that $z \in R_1[x]$ and show that $z \in h^{-1}[Y]$: As xR_1z, then, by M4, $h(x)R_2h(z)$, i.e. $h(z) \in R_2[h(x)] \subseteq Y$, hence $z \in h^{-1}[Y]$.

($\supseteq$) Assume that $z \in R_2[h(x)]$ and show that $z \in Y$: If $h(x)R_2z$, then, by M5', there exists $y \in R_1[x] \subseteq h^{-1}[Y]$ such that $h(y) \leq_2 z$. As $h(y) \in Y$ and Y is $\leq_2$-increasing, then $z \in Y$.

Let us show that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are general $\mathbf{IntK}_\Diamond$-frames, then for every $Y \in \mathcal{A}_2$,

$$h^{-1}[\Diamond_{R_2} Y] = \Diamond_{R_1} h^{-1}[Y].$$

For every $x \in X_1$, $x \in h^{-1}[\Diamond_{R_2} Y]$ iff $R_2[h(x)] \cap Y \neq \emptyset$, and $x \in \Diamond_{R_1} h^{-1}[Y]$ iff $R_1[x] \cap h^{-1}[Y] \neq \emptyset$.

($\subseteq$) Assume that $z \in R_2[h(x)] \cap Y$. As $h(x)R_2z$, then, by M5, there exists $y \in R_1[x]$ such that $z \leq_2 h(y)$. As $z \in Y$ and Y is $\leq_2$-increasing, then $h(y) \in Y$. Hence $y \in R_1[x] \cap h^{-1}[Y] \neq \emptyset$.

($\supseteq$) Assume that $z \in R_1[x] \cap h^{-1}[Y]$, hence $h(z) \in Y$ and xR_1z, so, by M4, $h(x)R_2h(z)$, i.e. $h(z) \in R_2[h(x)]$, and so $h(z) \in R_2[h(x)] \cap Y \neq \emptyset$.

Let us show that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are general $\mathbf{IK}$-frames, then for every $Y \in \mathcal{A}_2$,

$$h^{-1}[\Box_{(\leq \circ R_2)} Y] = \Box_{(\leq \circ R_1)} h^{-1}[Y].$$

For every $x \in X_1$, $x \in h^{-1}[\Box_{(\leq \circ R_2)} Y]$ iff $(\leq \circ R_2)[h(x)] \subseteq Y$, and $x \in \Box_{(\leq \circ R_1)} h^{-1}[Y]$ iff $(\leq \circ R_1)[x] \subseteq h^{-1}[Y]$.

($\subseteq$) Assume that $z \in (\leq \circ R_1)[x]$ and show that $z \in h^{-1}[Y]$: As $x \leq_1 wR_1z$ for some $w \in X_1$, then, by M1 and M4, $h(x) \leq_2 h(w)R_2h(z)$, i.e. $h(z) \in (\leq \circ R_2)[h(x)] \subseteq Y$, hence $z \in h^{-1}[Y]$.

($\supseteq$) Assume that $z \in (\leq \circ R_2)[h(x)]$ and show that $z \in Y$: If $h(x)(\leq_2 \circ R_2)z$, then, by M5, there exists $y \in (\leq_1 \circ R_1)[x] \subseteq h^{-1}[Y]$ such that $h(y) \leq_2 z$. As $h(y) \in Y$ and Y is $\leq_2$-increasing, then $z \in Y$.

The proof that $h^{-1}[\Diamond_{R_2} Y] = \Diamond_{R_1} h^{-1}[Y]$ goes as in the $\mathbf{IntK}_\Diamond$ case. $\square$

5 From algebras to general L-frames

Let $\mathbf{L} \in \{\mathbf{IntK}_\Box, \mathbf{IntK}_\Diamond, \mathbf{IK}\}$. For every $\mathbf{L}$-algebra $\mathcal{A}$ let $Pr(\mathcal{A})$ be the collection of the prime filters of $\mathcal{A}$. Let us define $\mathcal{A}_+ := \langle Pr(\mathcal{A}), \subseteq, \mathcal{R}_\mathcal{A}, \overline{\mathcal{A}} \rangle$, where for every $P, Q \in Pr(\mathcal{A})$:

R1. If $\mathcal{A}$ is an $\mathbf{IntK}_\Box$-algebra, $P \mathcal{R}_\mathcal{A} Q$ iff $\Box^{-1}[P] \subseteq Q$.

R2. If $\mathcal{A}$ is an $\mathbf{IntK}_\Diamond$-algebra, $P \mathcal{R}_\mathcal{A} Q$ iff $Q \subseteq \Diamond^{-1}[P]$.

R3. If $\mathcal{A}$ is an $\mathbf{IK}$-algebra, $P \mathcal{R}_\mathcal{A} Q$ iff $\Box^{-1} P \subseteq Q \subseteq \Diamond^{-1}[P]$.

$\overline{A} = \{\overline{a} \mid a \in \mathcal{A}\}$, and for every $a \in A$, $\overline{a} = \{P \in Pr(\mathcal{A}) \mid a \in P\}$, moreover for every n-ary operation $*$ in the signature of $\mathcal{A}$ $*^{\overline{\mathcal{A}}}(\overline{a_1}, \ldots, \overline{a_n}) = \overline{*(a_1, \ldots, a_n)}$.

For every homomorphism $f : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ let $f_+ : \mathcal{A}_{2+} \longrightarrow \mathcal{A}_{1+}$ be given by the assignment $P \longmapsto f^{-1}[P]$ for every $P \in Pr(\mathcal{A}_2)$.

5.1 Properties of $\mathcal{R}_\mathcal{A}$

Lemma 5.1.1. *For every* $\mathbf{L}$*-algebra* $\mathcal{A}$*,* $\mathcal{R}_\mathcal{A}$ *is a closed subset of* $\mathbf{X}_{\mathcal{A}_+} \times \mathbf{X}_{\mathcal{A}_+}$*.*

Proof. Assume that $\mathcal{R}_\mathcal{A}$ is defined like in R1. If $\langle P, Q \rangle \notin \mathcal{R}_\mathcal{A}$, then $\Box^{-1}[P] \not\subseteq Q$, i.e. $\Box a \in P$ and $a \notin Q$ for some $a \in \mathcal{A}$. Hence $P \in \overline{(\Box a)}$ and $Q \notin \overline{a}$. Let us consider $\mathcal{U} = \overline{(\Box a)} \times (Pr(\mathcal{A}) \setminus \overline{a})$. $\mathcal{U}$ is an open subset of $\mathbf{X}_{\mathcal{A}_+} \times \mathbf{X}_{\mathcal{A}_+}$, for both $\overline{(\Box a)}$ and $Pr(\mathcal{A}) \setminus \overline{a}$ are, moreover $\langle P, Q \rangle \in \mathcal{U}$. Let us show that $\mathcal{R}_\mathcal{A} \cap \mathcal{U} = \emptyset$: If $\langle S, T \rangle \in \mathcal{U}$, then $\Box a \in S$ and $a \notin T$, hence $\Box^{-1}[S] \not\subseteq T$, i.e. $\langle S, T \rangle \notin \mathcal{R}_\mathcal{A}$.

Assume that $\mathcal{R}_\mathcal{A}$ is defined like in R2. If $\langle P, Q \rangle \notin \mathcal{R}_\mathcal{A}$, then $Q \not\subseteq \Diamond^{-1}[P]$, i.e. $a \in Q$ and $\Diamond a \notin P$ for some $a \in \mathcal{A}$. Hence $Q \in \overline{a}$ and $P \notin \overline{(\Diamond a)}$. Let us consider $\mathcal{U} = (Pr(\mathcal{A}) \setminus \overline{(\Diamond a)}) \times \overline{a}$. $\mathcal{U}$ is an open subset of $\mathbf{X}_{\mathcal{A}_+} \times \mathbf{X}_{\mathcal{A}_+}$, for both $Pr(\mathcal{A}) \setminus \overline{(\Diamond a)}$ and $\overline{a}$ are, moreover $\langle P, Q \rangle \in \mathcal{U}$. Let us show that $\mathcal{R}_\mathcal{A} \cap \mathcal{U} = \emptyset$: If $\langle S, T \rangle \in \mathcal{U}$, then $\Diamond a \notin S$ and $a \in T$, hence $T \not\subseteq \Diamond^{-1}[S]$, i.e. $\langle S, T \rangle \notin \mathcal{R}_\mathcal{A}$.

Assume that $\mathcal{R}_\mathcal{A}$ is defined like in R3. If $\langle P, Q \rangle \notin \mathcal{R}_\mathcal{A}$, then either $\Box^{-1}[P] \not\subseteq Q$ or $Q \not\subseteq \Diamond^{-1}[P]$. Then the proof follows like in one of the cases above. $\square$

Corollary 5.1.2. *For every* $\mathbf{L}$*-algebra* $\mathcal{A}$*, if* $\mathcal{F}$ *is a closed subset of* $\mathbf{X}_{\mathcal{A}_+}$*, then* $\mathcal{R}_\mathcal{A}[\mathcal{F}]$ *is a closed subset of* $\mathbf{X}_{\mathcal{A}_+}$*.*

Proof. For every closed subset $\mathcal{F}$ of $\mathbf{X}_{\mathcal{A}_+}$,

$$\begin{aligned}\mathcal{R}_\mathcal{A}[\mathcal{F}] &= \{Q \in Pr(\mathcal{A}) \mid P \mathcal{R}_\mathcal{A} Q \text{ for some } P \in \mathcal{F}\} \\ &= \pi_2[\mathcal{R}_\mathcal{A} \cap (\mathcal{F} \times Pr(\mathcal{A}))].\end{aligned}$$

By 5.1.1 $\mathcal{R}_{\mathcal{A}}$ is closed, hence so is $\mathcal{R}_{\mathcal{A}} \cap (\mathcal{F} \times Pr(\mathcal{A}))$, and as π_2 is a closed map, for it is a continuous map between compact spaces, then $\pi_2[\mathcal{R}_{\mathcal{A}} \cap (\mathcal{F} \times Pr(\mathcal{A}))]$ is closed. □

Lemma 5.1.3. *1. For every* $\mathbf{IntK}_{\Box}$*-algebra* $\mathcal{A}$, $\mathcal{R}_{\mathcal{A}} = (\subseteq \circ \mathcal{R}_{\mathcal{A}} \circ \subseteq)$.

2. For every $\mathbf{IntK}_{\Diamond}$*-algebra* $\mathcal{A}$, $\mathcal{R}_{\mathcal{A}} = (\supseteq \circ \mathcal{R}_{\mathcal{A}} \circ \supseteq)$.

3. For every $\mathbf{IK}$*-algebra* $\mathcal{A}$, $\mathcal{R}_{\mathcal{A}} = (\subseteq \circ \mathcal{R}_{\mathcal{A}}) \cap (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$.

Proof. 1. ($\supseteq$) If $\langle P, Q\rangle \in (\subseteq \circ \mathcal{R}_{\mathcal{A}} \circ \subseteq)$, then $P \subseteq S_1 \mathcal{R}_{\mathcal{A}} S_2 \subseteq Q$, for some $S_1, S_2 \in Pr(\mathcal{A})$, hence $\Box^{-1}[P] \subseteq \Box^{-1}[S_1] \subseteq S_2 \subseteq Q$.
($\subseteq$) If $P\mathcal{R}_{\mathcal{A}}Q$, then $P \subseteq P\mathcal{R}_{\mathcal{A}}Q \subseteq Q$.
2.($\supseteq$) If $\langle P, Q\rangle \in (\supseteq \circ \mathcal{R}_{\mathcal{A}} \circ \supseteq)$, then $P \supseteq S_1 \mathcal{R}_{\mathcal{A}} S_2 \supseteq Q$, for some $S_1, S_2 \in Pr(\mathcal{A})$, hence $Q \subseteq S_2 \subseteq \Diamond^{-1}[S_1] \subseteq \Diamond^{-1}[P]$.
($\subseteq$) If $P\mathcal{R}_{\mathcal{A}}Q$, then $P \supseteq P\mathcal{R}_{\mathcal{A}}Q \supseteq Q$.
3. ($\supseteq$) If $\langle P, Q\rangle \in (\subseteq \circ \mathcal{R}_{\mathcal{A}}) \cap (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$, then $P \subseteq S_1\mathcal{R}_{\mathcal{A}}Q$ and $P\mathcal{R}_{\mathcal{A}}S_2 \supseteq Q$ for some $S_1, S_2 \in Pr(\mathcal{A})$, then $\Box^{-1}[P] \subseteq \Box^{-1}[S_1] \subseteq Q$ and $Q \subseteq S_2 \subseteq \Diamond^{-1}[P]$, hence $P\mathcal{R}_{\mathcal{A}}Q$.
($\subseteq$) If $P\mathcal{R}_{\mathcal{A}}Q$, then $P \subseteq P\mathcal{R}_{\mathcal{A}}Q$ and $P\mathcal{R}_{\mathcal{A}}Q \supseteq Q$. □

Lemma 5.1.4. *For every* $\mathbf{IK}$*-algebra* $\mathcal{A}$ *and every* $P, Q \in Pr(\mathcal{A})$,

1. $\langle P, Q\rangle \in (\mathcal{R}_{\mathcal{A}} \circ \supseteq)$ *iff* $Q \subseteq \Diamond^{-1}[P]$.
2. $\langle P, Q\rangle \in (\subseteq \circ \mathcal{R}_{\mathcal{A}})$ *iff* $\Box^{-1}[P] \subseteq Q$.

Proof. 1. ($\Leftarrow$) Assume that $Q \subseteq \Diamond^{-1}[P]$, and let us show that there exists $S \in Pr(\mathcal{A})$ such that $P\mathcal{R}_{\mathcal{A}}S \supseteq Q$, i.e. such that $Q \cup \Box^{-1}[P] \subseteq S$ and $S \cap \Diamond^{-1}[P]^c = \emptyset$. Let us consider $Fi(Q \cup \Box^{-1}[P])$: If we show that

$$Fi(Q \cup \Box^{-1}[P]) \cap \Diamond^{-1}[P]^c = \emptyset,$$

then the statement will follow by Birkhoff-Stone theorem. Suppose that $Fi(Q \cup \Box^{-1}[P]) \cap \Diamond^{-1}[P]^c \neq \emptyset$. Then there exists $c \in A$ such that $\Diamond c \notin P$ and $a \wedge b \leq c$ for some $a \in \Box^{-1}[P]$ and $b \in Q$. Then $b \leq a \to c$, hence $\Diamond b \leq \Diamond(a \to c) \leq (\Box a \to \Diamond c)$. As $b \in Q \subseteq \Diamond^{-1}[P]$, then $\Diamond b \in P$, hence $\Box a \to \Diamond c \in P$, and as $\Box a \in P$, then $\Diamond c \in P$, contradiction.
($\Rightarrow$) If $P\mathcal{R}_{\mathcal{A}}S \supseteq Q$ for some $S \in Pr(\mathcal{A})$, then $Q \subseteq S \subseteq \Diamond^{-1}[P]$.
2. ($\Leftarrow$) Assume that $\Box^{-1}[P] \subseteq Q$, and let us show that there exists $S \in Pr(\mathcal{A})$ such that $P \subseteq S$ and $\Box^{-1}[P] \subseteq Q \subseteq \Diamond^{-1}[P]$, i.e. such that $P \cup \Diamond[P] \subseteq S$ and $S \cap \Box[Q^c] = \emptyset$. Let us consider $Fi(P \cup \Diamond[Q])$: If we show that

$$Fi(P \cup \Diamond[Q]) \cap \Box[Q^c] = \emptyset,$$

then the statement will follow by Birkhoff-Stone theorem. Suppose that $Fi(P \cup \Diamond[Q]) \cap \Box[Q^c] \neq \emptyset$. Then there exist $a \in Q^c$, $b \in P$ and $c \in Q$ such that $b \wedge \Diamond c \leq \Box a$. Then $b \leq \Diamond c \to \Box a \leq \Box(c \to a)$. As $b \in P$, then $\Box(c \to a) \in P$, hence $c \to a \in \Box^{-1}[P] \subseteq Q$, and as $c \in Q$, then $a \in Q$, contradiction.
($\Rightarrow$) If $P \subseteq S\mathcal{R}_{\mathcal{A}}Q$ for some $S \in Pr(\mathcal{A})$, then $\Box^{-1}[P] \subseteq \Box^{-1}[S] \subseteq Q$. □

Corollary 5.1.5. *1. For every* $\mathbf{IntK}_\Box$*-algebra* $\mathcal{A}$ *and every* $P \in Pr(\mathcal{A})$*, if* $\Box a \notin P$*, then* $a \notin Q$ *and* $P\mathcal{R}_\mathcal{A}Q$ *for some* $Q \in Pr(\mathcal{A})$*.*

2. *For every* $\mathbf{IntK}_\Diamond$*-algebra* $\mathcal{A}$ *and every* $P \in Pr(\mathcal{A})$*, if* $\Diamond a \in P$*, then* $a \in Q$ *and* $P\mathcal{R}_\mathcal{A}Q$ *for some* $Q \in Pr(\mathcal{A})$*.*

3. *For every* $\mathbf{IK}$*-algebra* $\mathcal{A}$*, if* $\Box a \notin P$*, then* $a \notin Q$ *and* $P \subseteq S\mathcal{R}_\mathcal{A}Q$ *for some* $Q, S \in Pr(\mathcal{A})$*.*

4. *For every* $\mathbf{IK}$*-algebra* $\mathcal{A}$ *and every* $P \in Pr(\mathcal{A})$*, if* $\Diamond a \in P$*, then* $a \in S$ *and* $P\mathcal{R}_\mathcal{A}S$ *for some* $S \in Pr(\mathcal{A})$*.*

Proof. 1. If $\Box a \notin P$, then $Id(a) \cap \Box^{-1}[P] = \emptyset$, for if not, then $c \leq a$ for some c such that $\Box c \in P$, hence $\Box c \leq \Box a$, therefore $\Box a \in P$, contradiction. By Birkhoff-Stone theorem, there exists $Q \in Pr(\mathcal{A})$ such that $\Box^{-1}[P] \subseteq Q$, i.e. $P\mathcal{R}_\mathcal{A}Q$, and $a \notin Q$.
2. If $\Diamond a \in P$, then $Fi(a) \cap \Diamond^{-1}[P^c] = \emptyset$, for if not, then $a \leq c$ for some c such that $\Diamond c \notin P$, hence $\Diamond a \leq \Diamond c$, therefore $\Diamond c \in P$, contradiction. By Birkhoff-Stone theorem, there exists $Q \in Pr(\mathcal{A})$ such that $a \in Q$ and $Q \subseteq \Diamond^{-1}[P]$, i.e. $P\mathcal{R}_\mathcal{A}Q$.
3. If $\Box a \notin P$, then $Id(a) \cap \Box^{-1}[P] = \emptyset$, so by Birkhoff-Stone theorem, there exists $Q \in Pr(\mathcal{A})$ such that $\Box^{-1}[P] \subseteq Q$ and $a \notin Q$. By item 2 of 5.1.4, $P \subseteq S\mathcal{R}_\mathcal{A}Q$ for some $S \in Pr(\mathcal{A})$.
4. If $\Diamond a \in P$, then $Fi(a) \cap \Diamond^{-1}[P^c] = \emptyset$, so by Birkhoff-Stone theorem, there exists $Q \in Pr(\mathcal{A})$ such that $a \in Q$ and $Q \subseteq \Diamond^{-1}[P]$. By item 1 of 5.1.4, $P\mathcal{R}_\mathcal{A}S \supseteq Q$ for some $S \in Pr(\mathcal{A})$, and as $a \in S \subseteq Q$, then $a \in S$. □

Corollary 5.1.6. *1. For every* $\mathbf{IntK}_\Box$*-algebra* $\mathcal{A}$*,* $(\subseteq \circ \mathcal{R}_\mathcal{A}) \subseteq (\mathcal{R}_\mathcal{A} \circ \subseteq)$*.*

2. *For every* $\mathbf{IntK}_\Diamond$*-algebra* $\mathcal{A}$*,* $(\supseteq \circ \mathcal{R}_\mathcal{A}) \subseteq (\mathcal{R}_\mathcal{A} \circ \supseteq)$*.*

3. *For every* $\mathbf{IK}$*-algebra* $\mathcal{A}$*,* $(\supseteq \circ \mathcal{R}_\mathcal{A}) \subseteq (\mathcal{R}_\mathcal{A} \circ \supseteq)$ *and* $(\mathcal{R}_\mathcal{A} \circ \subseteq) \subseteq (\subseteq \circ \mathcal{R}_\mathcal{A})$*.*

Proof. 1. If $P \subseteq S\mathcal{R}_\mathcal{A}Q$, then $\Box^{-1}[P] \subseteq \Box^{-1}[S] \subseteq Q$, hence $P\mathcal{R}_\mathcal{A}Q \subseteq Q$.
2. If $P \supseteq S\mathcal{R}_\mathcal{A}Q$, then $Q \subseteq \Diamond^{-1}[S] \subseteq \Diamond^{-1}[P]$, hence $P\mathcal{R}_\mathcal{A}Q \supseteq Q$.
3. If $P\mathcal{R}_\mathcal{A}S \subseteq Q$, then $\Box^{-1}[P] \subseteq S \subseteq Q$, hence by item 2 of 5.1.4, $\langle P, Q\rangle \in (\subseteq \circ \mathcal{R}_\mathcal{A})$.
If $P \supseteq S\mathcal{R}_\mathcal{A}Q$, then $Q \subseteq \Diamond^{-1}[S] \subseteq \Diamond^{-1}[P]$, hence hence by item 1 of 5.1.4, $\langle P, Q\rangle \in (\mathcal{R}_\mathcal{A} \circ \supseteq)$. □

5.2 The action of $(\_)_+$ on objects

Proposition 5.2.1. *Let* $\mathbf{L} \in \{\mathbf{IntK}_\Box, \mathbf{IntK}_\Diamond, \mathbf{IK}\}$*. For every* $\mathbf{L}$*-algebra* $\mathcal{A}$*,* $\mathcal{A}_+ = \langle Pr(\mathcal{A}), \subseteq, \mathcal{R}_\mathcal{A}, \overline{\mathcal{A}}\rangle$ *is a general* $\mathbf{L}$*-frame.*

Proof. From the duality for Heyting algebras, it holds that $\mathbf{X}_{\mathcal{A}_+}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen increasing sets of $\mathbf{X}_{\mathcal{A}_+}$, which is D1. As $\mathbf{X}_{\mathcal{A}_+}$ is an Esakia space, then in particular it is Hausdorff, hence for every

$P \in Pr(\mathcal{A})$, $\{P\}$ is closed in $\mathbf{X}_{\mathcal{A}_+}$, and so by 5.1.2, $\mathcal{R}_{\mathcal{A}}[P]$ is closed in $\mathbf{X}_{\mathcal{A}_+}$, which is D3.

Let us show that if $\mathcal{A}$ is an $\mathbf{IntK}_\Box$-algebra, then $\Box^{\overline{\mathcal{A}}} = \Box_{\mathcal{R}_\mathcal{A}}$, i.e. that for every $a \in \mathcal{A}$,

$$\overline{(\Box a)} = \Box_{\mathcal{R}_\mathcal{A}} \overline{a}.$$

($\subseteq$) If $P \in \overline{(\Box a)}$, then $\Box a \in P$, i.e. $a \in \Box^{-1}[P]$ so, for every $Q \in Pr(\mathcal{A})$, if $P\mathcal{R}_\mathcal{A}Q$, then $a \in \Box^{-1}[P] \subseteq Q$.

($\supseteq$) If $P \notin \overline{(\Box a)}$, then by item 1 of 5.1.5, $a \notin Q$ and $P\mathcal{R}_\mathcal{A}Q$ for some $Q \in Pr(\mathcal{A})$, so $P \notin \Box_{\mathcal{R}_\mathcal{A}} \overline{a}$.

Let us show that if $\mathcal{A}$ is an $\mathbf{IntK}_\Diamond$-algebra (an $\mathbf{IK}$-algebra), then $\Diamond^{\overline{\mathcal{A}}} = \Diamond_{\mathcal{R}_\mathcal{A}}$, i.e. that for every $a \in \mathcal{A}$,

$$\overline{(\Diamond a)} = \Diamond_{\mathcal{R}_\mathcal{A}} \overline{a}.$$

($\subseteq$) If $P \in \overline{(\Diamond a)}$, then $\Diamond a \in P$, then by item 2 (item 4) of 5.1.5, $a \in Q$ and $P\mathcal{R}_\mathcal{A}Q$ for some $Q \in Pr(\mathcal{A})$, hence $P \in \Diamond_{\mathcal{R}_\mathcal{A}} \overline{a}$.

($\supseteq$) If $P \in \Diamond_{\mathcal{R}_\mathcal{A}} \overline{a}$, then $a \in Q$ and $P\mathcal{R}_\mathcal{A}Q$ for some $Q \in Pr(\mathcal{A})$, i.e. $Q \subseteq \Diamond^{-1}[P]$, hence $\Diamond a \in P$.

Let us show that if $\mathcal{A}$ is an $\mathbf{IK}$-algebra, then $\Box^{\overline{\mathcal{A}}} = \Box_{(\subseteq \circ \mathcal{R}_\mathcal{A})}$, i.e. that for every $a \in \mathcal{A}$,

$$\overline{(\Box a)} = \Box_{(\subseteq \circ \mathcal{R}_\mathcal{A})} \overline{a}.$$

($\subseteq$) If $P \in \overline{(\Box a)}$, then $\Box a \in P$, i.e. $a \in \Box^{-1}[P]$ so, for every $Q \in Pr(\mathcal{A})$, if $P\mathcal{R}_\mathcal{A}Q$, then $a \in \Box^{-1}[P] \subseteq Q$.

($\supseteq$) If $P \notin \overline{(\Box a)}$, then by item 3 of 5.1.5, $a \notin Q$ and $P \subseteq S\mathcal{R}_\mathcal{A}Q$ for some $Q, S \in Pr(\mathcal{A})$, so $P \notin \Box_{(\subseteq \circ \mathcal{R}_\mathcal{A})} \overline{a}$.

This is enough to show that $\overline{\mathcal{A}}$ is closed in each case under the appropriate operations.

If $\mathcal{A}$ is an $\mathbf{IK}$-algebra, then as $\mathbf{X}_{\mathcal{A}_+}$ is an Esakia space, then in particular it is Priestley, hence for every $P \in Pr(\mathcal{A})$, $P{\uparrow} = \{Q \in Pr(\mathcal{A}) \mid P \subseteq Q\}$ is closed in $\mathbf{X}_{\mathcal{A}_+}$, and so by 5.1.2, $\mathcal{R}_\mathcal{A}[P{\uparrow}]$ is closed in $\mathbf{X}_{\mathcal{A}_+}$.

Let us show that $\mathcal{R}_\mathcal{A}[P{\uparrow}]$ is $\subseteq$-increasing: If $Q \in \mathcal{R}_\mathcal{A}[P{\uparrow}]$ and $Q \subseteq T$, then $P \subseteq S\mathcal{R}_\mathcal{A}Q \subseteq T$, hence by item 3 of 5.1.6, $P \subseteq S \subseteq Q'\mathcal{R}_\mathcal{A}T$, and so $T \in \mathcal{R}_\mathcal{A}[P{\uparrow}]$. This proves D4. $\Box$

5.3 The action of $(\_)_+$ on arrows

Proposition 5.3.1. *Let* $\mathbf{L} \in \{\mathbf{IntK}_\Box, \mathbf{IntK}_\Diamond, \mathbf{IK}\}$. *For every* $\mathbf{L}$*-algebra homomorphism* $h : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$*,* $h_+ : \mathcal{A}_{2+} \longrightarrow \mathcal{A}_{1+}$ *is a p-morphism of general* $\mathbf{L}$*-frames.*

Proof. From the duality for Heyting algebras, it holds that h_+ is a continuous and strongly isotone map between $\mathbf{X}_{\mathcal{A}_{2+}}$ and $\mathbf{X}_{\mathcal{A}_{1+}}$, which is equivalent to conditions M1–M3.

Let us show that if $P, Q \in Pr(\mathcal{A}_2)$ and $\Box^{-1}[P] \subseteq Q$, then $\Box^{-1}[h^{-1}[P]] \subseteq h^{-1}[Q]$: For every $a \in \mathcal{A}_2$,

$$\begin{array}{rcl} a \in \Box^{-1}[h^{-1}[P]] & \Leftrightarrow & \Box a \in h^{-1}[P] \\ & \Leftrightarrow & h(\Box a) \in P \\ & \Leftrightarrow & \Box h(a) \in P \\ & \Leftrightarrow & h(a) \in \Box^{-1}[P] \subseteq Q \\ & \Rightarrow & a \in h^{-1}[Q]. \end{array}$$

Let us show that if $P, Q \in Pr(\mathcal{A}_2)$ and $Q \subseteq \Diamond^{-1}[P]$, then $h^{-1}[Q] \subseteq \Diamond^{-1}[h^{-1}[P]]$: For every $a \in \mathcal{A}_2$,

$$\begin{array}{rcl} a \in h^{-1}[Q] & \Leftrightarrow & h(a) \in Q \subseteq \Diamond^{-1}[P] \\ & \Rightarrow & \Diamond h(a) \in P \\ & \Leftrightarrow & h(\Diamond a) \in P \\ & \Leftrightarrow & \Diamond a \in h^{-1}[P] \\ & \Leftrightarrow & a \in \Diamond^{-1}[h^{-1}[P]]. \end{array}$$

This is enough to show that for $\mathbf{L} \in \{\mathbf{IntK}_\Box, \mathbf{IntK}_\Diamond, \mathbf{IK}\}$ and for every $P, Q \in Pr(\mathcal{A}_2)$, if $P\mathcal{R}_{\mathcal{A}_2}Q$, then $h_+(P)\mathcal{R}_{\mathcal{A}_1}h_+(Q)$, which is M4.

Let us show M5 for $\mathbf{IntK}_\Diamond$-algebras, i.e. that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathbf{IntK}_\Diamond$-algebras and $P \in Pr(\mathcal{A}_2)$, $Q \in Pr(\mathcal{A}_1)$ are such that $h^{-1}[P]\mathcal{R}_{\mathcal{A}_1}Q$, then there exists $S \in \mathcal{R}_{\mathcal{A}_2}[P]$ such that $Q \subseteq h^{-1}[S]$. We need that $S \subseteq \Diamond^{-1}[P]$, i.e. $S \cap \Diamond^{-1}[P]^c = \emptyset$ and $Q \subseteq h^{-1}[S]$, i.e. $h[Q] \subseteq S$. It holds that

$$Fi(h[Q]) \cap \Diamond^{-1}[P]^c = \emptyset,$$

for if not, then there are $a \in Q$ and $\Diamond b \notin P$ such that $h(a) \leq b$, hence $\Diamond h(a) \leq \Diamond b$. As $a \in Q \subseteq \Diamond^{-1}[h^{-1}[P]]$, then $\Diamond h(a) \in P$, hence $\Diamond b \in P$, contradiction.

By Birkhoff-Stone theorem, there exists $S \in Pr(\mathcal{A}_2)$ such that $h[Q] \subseteq S$ (i.e. $Q \subseteq h^{-1}[S]$) and $S \cap \Diamond^{-1}[P]^c = \emptyset$, i.e. $S \subseteq \Diamond^{-1}[P]$, i.e. $P\mathcal{R}_{\mathcal{A}_2}S$.

Let us show M5 for **IK**-algebras: Like before, it holds that $Fi(h[Q]) \cap \Diamond^{-1}[P]^c = \emptyset$, so by Birkhoff-Stone theorem, $h[Q] \subseteq T$ (i.e. $Q \subseteq h^{-1}[T]$) and $T \cap \Diamond^{-1}[P]^c = \emptyset$ for some $T \in Pr(\mathcal{A}_2)$. As $T \subseteq \Diamond^{-1}[P]$, then by item 1 of 5.1.4, $\langle P, T\rangle \in (\mathcal{R}_{\mathcal{A}_2} \circ \supseteq)$, i.e. $P\mathcal{R}_{\mathcal{A}_2}S \supseteq T$ for some $S \in Pr(\mathcal{A}_2)$, so $S \in \mathcal{R}_{\mathcal{A}_2}[P]$ and $Q \subseteq h^{-1}[T] \subseteq h^{-1}[S]$.

Let us show M5', i.e. that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathbf{IntK}_\Box$-algebras and $P \in Pr(\mathcal{A}_2)$, $Q \in Pr(\mathcal{A}_1)$ are such that $h^{-1}[P]\mathcal{R}_{\mathcal{A}_1}Q$, then there exists $S \in \mathcal{R}_{\mathcal{A}_2}[P]$ such that $h^{-1}[S] \subseteq Q$: we need that $\Box^{-1}[P] \subseteq S$ and $S \subseteq h[Q]$, i.e. $S \cap h[Q^c] = \emptyset$. If we show that

$$h[Q^c] \cap Fi(\Box^{-1}[P]) = \emptyset,$$

then the statement will follow from Birkhoff-Stone theorem. Suppose that there are $a \notin Q$ and $\Box b \in P$ such that $b \leq h(a)$, hence $\Box b \leq \Box h(a) = h(\Box a)$. As $\Box b \in P$, then $h(\Box a) \in P$, hence $a \in \Box^{-1}[h^{-1}[P] \subseteq Q$, contradiction.

Let us show M6, i.e. that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are **IK**-algebras and $P \in Pr(\mathcal{A}_2)$, $Q \in Pr(\mathcal{A}_1)$ are such that $h^{-1}[P](\subseteq \circ \mathcal{R}_{\mathcal{A}_1})Q$, then there exists $S \in (\subseteq \circ \mathcal{R}_{\mathcal{A}_1})[P]$ such that $h^{-1}[S] \subseteq Q$: By item 2 of 5.1.4, we need that $\Box^{-1}[P] \subseteq S$, moreover, we need that $S \subseteq h[Q]$, i.e. $S \cap h[Q^c] = \emptyset$. The proof goes like in the case treated before. $\Box$

6 Duality

For $\mathbf{L} \in \{\mathbf{IntK}_\Box, \mathbf{IntK}_\Diamond, \mathbf{IK}\}$ and for every general $\mathbf{L}$-frame $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$, let us consider the assignment which maps every $x \in X$ to the set $\varepsilon_\mathcal{G}(x) = \{Y \in \mathcal{A} \mid x \in Y\}$. From the duality for Heyting algebras, we know that this assignment defines a map $\varepsilon_\mathcal{G} : \mathbf{X}_\mathcal{G} \longrightarrow \mathbf{X}_{(\mathcal{G}^+)_+}$ which is an iso in $\mathbf{E}$.
Let us introduce three full subcategories of the categories of the general $\mathbf{L}$-frames and their p-morphisms:

6.1 L-spaces

Definition 6.1.1. **($\mathbf{IntK}_\Box$-space)** *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is an* $\mathbf{IntK}_\Box$-space *iff*

D1. *$\mathbf{X}_\mathcal{G}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen increasing sets of $\mathbf{X}_\mathcal{G}$.*

D2'. *$\mathcal{A}$ is closed under $\Box_R$.*

D3. *For every $x \in X$, $R[x] \in K^\uparrow(\mathbf{X}_\mathcal{G}) = \{F \in K(\mathbf{X}_\mathcal{G}) \mid F = F\uparrow\}$.*

So $\mathbf{IntK}_\Box$-spaces are those general $\mathbf{IntK}_\Box$-frames such that $R[x]$ is $\leq$-increasing for every $x \in X$. Let $\mathbf{IntK}_\Box\mathbf{sp}$ be the category of the $\mathbf{IntK}_\Box$-spaces and their p-morphisms.

Example 6.1.2. *For every finite partial order $\langle X, \leq\rangle$, the general frame $\mathcal{G} = \langle X, \leq, \leq, \mathcal{P}_\leq(X)\rangle$ is an $\mathbf{IntK}_\Box$-space.*

Proof. Let τ be the topology generated by taking $\mathcal{P}_\leq(X) \cup \mathcal{P}_\geq(X)$ as a subbase, and let $\mathbf{X} = \langle X, \leq, \tau\rangle$. In 3.3.2, we saw that $\mathbf{X}$ is an Esakia space and that $\mathcal{P}_\leq(X)$ is the collection of the clopen increasing subsets of $\mathbf{X}$. Item 1 of 2.0.8 implies that $\mathcal{P}_\leq(X)$ is closed under $\Box_\leq$. For every $x \in X$, $x\uparrow \in \mathcal{P}_\leq(X)$ is clopen increasing, so in particular $x\uparrow \in K^\uparrow(\mathbf{X})$. □

Definition 6.1.3. **($\mathbf{IntK}_\Diamond$-space)** *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ be a general frame. $\mathcal{G}$ is an* $\mathbf{IntK}_\Diamond$-space *iff*

D1. *$\mathbf{X}_\mathcal{G}$ is an Esakia space, and $\mathcal{A}$ is the collection of the clopen increasing sets of $\mathbf{X}_\mathcal{G}$.*

D2. *$\mathcal{A}$ is closed under $\Diamond_R$.*

D3. *For every $x \in X$, $R[x] \in K^\downarrow(\mathbf{X}_\mathcal{G}) = \{F \in K(\mathbf{X}_\mathcal{G}) \mid F = F\downarrow\}$.*

So $\mathbf{IntK}_\Diamond$-spaces are those general $\mathbf{IntK}_\Diamond$-frames such that $R[x]$ is $\leq$-decreasing for every $x \in X$. Let $\mathbf{IntK}_\Diamond\mathbf{sp}$ be the category of the $\mathbf{IntK}_\Diamond$-spaces and their p-morphisms.

Example 6.1.4. *For every finite partial order $\langle X, \leq\rangle$, the general frame $\mathcal{G} = \langle X, \leq, \geq, \mathcal{P}_\leq(X)\rangle$ is an $\mathbf{IntK}_\Diamond$-space.*

Proof. Let τ be the topology generated by taking $\mathcal{P}_{\leq}(X) \cup \mathcal{P}_{\geq}(X)$ as a subbase, and let $\mathbf{X} = \langle X, \leq, \tau \rangle$. In 3.3.2, we saw that $\mathbf{X}$ is an Esakia space and that $\mathcal{P}_{\leq}(X)$ is the collection of the clopen increasing subsets of $\mathbf{X}$. Item 2 of 2.0.8 implies that $\mathcal{P}_{\leq}(X)$ is closed under $\Diamond_{\geq}$. For every $x \in X$, $x{\downarrow} \in \mathcal{P}_{\geq}(X)$ is clopen decreasing, so in particular $x{\downarrow} \in K^{\downarrow}(\mathbf{X})$. □

Definition 6.1.5. (IK-space) *Let $\mathcal{G} = \langle X, \leq, R, \mathcal{A} \rangle$ be a general frame. $\mathcal{G}$ is an* **IK**-space *iff*

D1. $\mathbf{X}_{\mathcal{G}}$ *is an Esakia space, and $\mathcal{A}$ is the collection of the clopen increasing sets of* $\mathbf{X}_{\mathcal{G}}$.

D2. $\mathcal{A}$ *is closed under* $\Diamond_R$ *and* $\Box_{(\leq \circ R)}$.

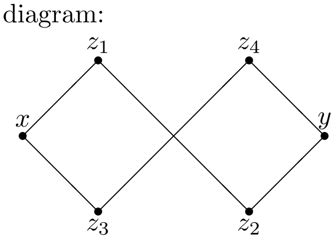
D3. For every $x \in X$, $R[x] \in K(\mathbf{X}_{\mathcal{G}})$.

D4. For every $x \in X$, $R[x{\uparrow}] \in K^{\uparrow}(\mathbf{X}_{\mathcal{G}})$.

D5. For every $x \in X$, $R[x] = R[x{\uparrow}] \cap R[x]{\downarrow}$.

Conditions D4 and D5 together imply that for every $x \in X$, $R[x]$ is the intersection of an increasing set and a decreasing set, hence $R[x]$ is convex, and so $R[x] = R[x]{\uparrow} \cap R[x]{\downarrow}$. So if $\mathcal{G}$ is an **IK**-space, then $\mathcal{G}$ is a general **IK**-space and $R[x]$ is convex for every $x \in X$. `Question: does the viceversa hold? Probably not.` Let **IKsp** be the category of the **IK**-spaces and their p-morphisms.

Given a finite partial order $\langle X, \leq \rangle$, the general **IK**-frame $\mathcal{G} = \langle X, \leq, (\geq \circ \leq), \mathcal{P}_{\leq}(X) \rangle$ is not an **IK**-space in general. Consider the partial order associated with the following Hasse diagram:



The relation $(\geq \circ \leq)$ does not satisfy D5: It holds that $x \leq z_1 \geq z_2 \leq y$, so $y \in (\geq \circ \leq)[x{\uparrow}]$, and $x \geq z_3 \leq z_4 \geq y$, so $y \in (\geq \circ \leq)[x]{\downarrow}$, but $y \notin (\geq \circ \leq)[x]$.

Example 6.1.6. *For every finite linear order $\langle X, \leq \rangle$, the general* **IK**-*frame $\mathcal{G} = \langle X, \leq, (\geq \circ \leq), \mathcal{P}_{\leq}(X) \rangle$ is an* **IK**-*space.*

Proof. Since $\leq$ is a linear order, then for every $x \in X$, $X = x{\uparrow} \cup x{\downarrow} \subseteq (\geq \circ \leq)[x]$, hence $(\geq \circ \leq)[x] = (\geq \circ \leq)[x{\uparrow}] \cap (\geq \circ \leq)[x]{\downarrow}$, which is D5. □

Proposition 6.1.7. *For every* **L**-*algebra $\mathcal{A}$, $\mathcal{A}_+$ is an* **L**-*space.*

Proof. By 5.2.1, $\mathcal{A}_+$ is a general **L**-frame. By item 1 of 5.1.3, if $\mathcal{A}$ is an $\mathbf{IntK}_\Box$-algebra, then $\mathcal{R}_\mathcal{A}[P] = (\subseteq \circ \mathcal{R}_\mathcal{A} \circ \subseteq)[P]$ is $\subseteq$-increasing for every $P \in Pr(\mathcal{A})$. Analogously, items 2 and 3 of 5.1.3 respectively imply that if $\mathcal{A}$ is an $\mathbf{IntK}_\Diamond$-algebra, then $\mathcal{R}_\mathcal{A}[P]$ is $\subseteq$-decreasing for every $P \in Pr(\mathcal{A})$, and if $\mathcal{A}$ is an **IK**-algebra, then $\mathcal{R}_\mathcal{A}[P] = (\subseteq \circ \mathcal{R}_\mathcal{A})[P] \cap (\mathcal{R}_\mathcal{A} \circ \supseteq)[P]$ for every $P \in Pr(\mathcal{A})$. □

Lemma 6.1.8. *For every general* **L***-frame* $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ *and every* $x, y \in X$,

1. $x \leq y$ *iff* $\varepsilon_\mathcal{G}(x) \subseteq \varepsilon_\mathcal{G}(y)$.
2. *If* xRy *then* $\varepsilon_\mathcal{G}(x)\mathcal{R}_\mathcal{A}\varepsilon_\mathcal{G}(y)$.

Proof. 1. If $x \leq y$ then, as $\mathcal{A} \subseteq \mathcal{P}_\leq(X)$, for every $Y \in \mathcal{A}$, if $x \in Y$ then $y \in Y$.
If $x \not\leq y$ then, as $\mathbf{X}_\mathcal{G}$ is totally order-disconnected and $\mathcal{A}$ is the collection of the clopen increasing subsets of $\mathbf{X}_\mathcal{G}$, $x \in Y$ and $y \notin Y$ for some $Y \in \mathcal{A}$, hence $Y \in (\varepsilon_\mathcal{G}(x) \setminus \varepsilon_\mathcal{G}(y))$, and so $\varepsilon_\mathcal{G}(x) \not\subseteq \varepsilon_\mathcal{G}(y)$.
2. Let us show that if $y \in R[x]$, then a) $\Box_R^{-1}[\varepsilon_\mathcal{G}(x)] \subseteq \varepsilon_\mathcal{G}(y)$, b) $\varepsilon_\mathcal{G}(y) \subseteq \Diamond_R^{-1}[\varepsilon_\mathcal{G}(x)]$ and c) $\Box_{(\leq\circ R)}^{-1}[\varepsilon_\mathcal{G}(x)] \subseteq \varepsilon_\mathcal{G}(y)$:
a) For every $Y \in \mathcal{A}$, $\Box_R Y \in \varepsilon_\mathcal{G}(x)$ iff $x \in \Box_R Y$, iff $R[x] \subseteq Y$, and so $y \in Y$, i.e. $Y \in \varepsilon_\mathcal{G}(y)$.
b) For every $Y \in \mathcal{A}$, $Y \in \varepsilon_\mathcal{G}(y)$ iff $y \in Y$, and as $y \in R[x]$, then $R[x] \cap Y \neq \emptyset$, i.e. $\Diamond_R Y \in \varepsilon_\mathcal{G}(x)$, i.e. $Y \in \Diamond_R^{-1}[\varepsilon_\mathcal{G}(x)]$.
c) For every $Y \in \mathcal{A}$, $\Box_{(\leq\circ R)} Y \in \varepsilon_\mathcal{G}(x)$ iff $x \in \Box_{(\leq\circ R)} Y$, iff $(\leq \circ R)[x] \subseteq Y$, and so $y \in R[x] \subseteq (\leq \circ R)[x] \subseteq Y$, i.e. $Y \in \varepsilon_\mathcal{G}(y)$.
a) proves the statement if $\mathcal{A}$ is an $\mathbf{IntK}_\Box$-algebra, b) proves the statement if $\mathcal{A}$ is an $\mathbf{IntK}_\Diamond$-algebra, and a) and c) together prove the statement if $\mathcal{A}$ is an **IK**-algebra. □

Lemma 6.1.9. 1. *The following are equivalent for every general* $\mathbf{IntK}_\Box$*-frame:*

   (a) *For every* $x \in X$, $R[x] = R[x]{\uparrow}$.
   (b) *For every* $x, y \in X$, *if* $\varepsilon(x)\mathcal{R}_\mathcal{A}\varepsilon(y)$ *then* xRy.

2. *The following are equivalent for every general* $\mathbf{IntK}_\Diamond$*-frame:*

   (a) *For every* $x \in X$, $R[x] = R[x]{\downarrow}$.
   (b) *For every* $x, y \in X$, *if* $\varepsilon(x)\mathcal{R}_\mathcal{A}\varepsilon(y)$ *then* xRy.

3. *The following are equivalent for every general* **IK***-frame:*

   (a) *For every* $x \in X$, $R[x] = R[x{\uparrow}] \cap R[x]{\downarrow}$.
   (b) *For every* $x, y \in X$, *if* $\varepsilon(x)\mathcal{R}_\mathcal{A}\varepsilon(y)$ *then* xRy.

Proof. 1. (a $\Rightarrow$ b) Suppose that $x, y \in X$ are such that $\varepsilon(x)\mathcal{R}_\mathcal{A}\varepsilon(y)$ but $y \notin R[x] = R[x]{\uparrow}$. Then $R[x] \subseteq U$ and $y \notin U$ for some $U \in \mathcal{A}$, hence $x \in \Box_R U$.
As $\varepsilon(x)\mathcal{R}_\mathcal{A}\varepsilon(y)$, then $\Box_R^{-1}[\varepsilon(x)] \subseteq \varepsilon(y)$, i.e. for every $U \in \mathcal{A}$, if $x \in \Box_R U$, then $y \in U$, contradiction.

(b $\Rightarrow$ a) ($\supseteq$) If $y \in R[x]\uparrow$, then $xRz \leq y$ for some $z \in X$, hence, by 6.1.8, $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(z) \subseteq \varepsilon(y)$, i.e. $\Box_R^{-1}[\varepsilon(x)] \subseteq \varepsilon(z) \subseteq \varepsilon(y)$, hence $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, and so by assumption it follows that xRy.

2. (a $\Rightarrow$ b) Suppose that $x, y \in X$ are such that $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$ but $y \notin R[x] = R[x]\downarrow$. Then $y \in U$ and $R[x] \cap U = \emptyset$ for some clopen increasing subset U, hence $x \notin \Diamond_R U$.

As $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, then $\varepsilon(y) \subseteq \Diamond_R^{-1}[\varepsilon(x)]$, i.e. for every $U \in \mathcal{A}$, if $y \in U$ then $x \in \Diamond_R U$, contradiction.

(b $\Rightarrow$ a) ($\supseteq$) If $y \in R[x]\downarrow$, then $xRz \geq y$ for some $z \in X$, hence, by 6.1.8, $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(z) \supseteq \varepsilon(y)$, i.e. $\varepsilon(y) \subseteq \varepsilon(z) \subseteq \Diamond_R^{-1}[\varepsilon(x)]$, hence $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, and so by assumption it follows that xRy.

3. (a $\Rightarrow$ b) Suppose that $x, y \in X$ are such that $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$ but $y \notin R[x] = R[x\uparrow] \cap R[x]\downarrow$. Then either $y \notin R[x\uparrow]$ or $y \notin R[x]\downarrow$. If $y \notin R[x\uparrow] = R[x\uparrow]\uparrow$ Then $R[x\uparrow] \subseteq U$ and $y \notin U$ for some $U \in \mathcal{A}$, hence $x \in \Box_{(\leq \circ R)}U$.

As $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, then $\Box_{(\leq \circ R)}^{-1}[\varepsilon(x)] \subseteq \varepsilon(y)$, i.e. for every $U \in \mathcal{A}$, if $x \in \Box_{(\leq \circ R)}U$, then $y \in U$, contradiction. If $y \notin R[x]\downarrow$ the proof is analogous to the (a $\Rightarrow$ b) of item 2.

(b $\Rightarrow$ a) ($\supseteq$) If $y \in R[x\uparrow] \cap R[x]\downarrow$, then $x \leq z_1 Ry$ and $xRz_2 \geq y$ for some $z_1, z_2 \in X$, hence, by 6.1.8, $\varepsilon(x) \subseteq \varepsilon(z_1)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$ and $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(z_2) \supseteq \varepsilon(y)$, and so $\Box_R^{-1}[\varepsilon(x)] \subseteq \Box_R^{-1}[\varepsilon(z_1)] \subseteq \varepsilon(y)$ and $\varepsilon(y) \subseteq \varepsilon(z_2) \subseteq \Diamond_R^{-1}[\varepsilon(x)]$, hence $\varepsilon(x)\mathcal{R}_{\mathcal{A}}\varepsilon(y)$, and so by assumption it follows that xRy. □

Proposition 6.1.10. *For every* **L***-space* $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$*,* $\varepsilon_{\mathcal{G}} : \mathcal{G} \longrightarrow (\mathcal{G}^+)_+$ *is a p-morphism of* **L***-spaces, hence it is an iso in the category of* **L***-spaces.*

Proof. From the duality for Heyting algebras, we know that $\varepsilon_{\mathcal{G}} : \mathbf{X}_{\mathcal{G}} \longrightarrow \mathbf{X}_{(\mathcal{G}^+)_+}$ is an iso in **E**, hence it is bijective and satisfies M1–M3. M4 holds by item 2 of 6.1.8. The surjectivity of $\varepsilon_{\mathcal{G}}$ and 6.1.9 imply M5', M5 and M6. Let us show M6: If $\varepsilon_{\mathcal{G}}(x)(\subseteq \circ \mathcal{R}_{\mathcal{A}})P = \varepsilon_{\mathcal{G}}(y)$, then $\varepsilon_{\mathcal{G}}(x) \subseteq \varepsilon_{\mathcal{G}}(z)\mathcal{R}_{\mathcal{A}}\varepsilon_{\mathcal{G}}(y)$ for some $z \in X$, hence, by item 1 of 6.1.8 and 6.1.9, $x \leq zRy$, i.e. $y \in (\leq \circ R)[x]$. □

Theorem 6.1.11. *For every* $\mathbf{L} \in \{\mathbf{IntK}_\Box, \mathbf{IntK}_\Diamond, \mathbf{IK}\}$*, the category* **LAlg** *of* **L***-algebras and their homomorphisms is dually equivalent to the category* **LSp** *of* **L***-spaces and their p-morphisms.*

Proof. It follows from 4.1.2, 4.2.1, 5.3.1, 6.1.7, and 6.1.10. □

7 Characterizing topological semantics of MIPC

One of the best known axiomatic extensions of **IK** is the *modal imtuitionistic propositional calculus* (**MIPC**) introduced by Prior in [12]. **MIPC** can be thought of as the intuitionistic S5 (see [2]), and it holds (see for example [13]) that

$$\begin{aligned}\mathbf{MIPC} \quad = \quad & \mathbf{IK} \oplus \Box p \to p \;\oplus \Box p \to \Box\Box p \;\oplus \Diamond p \to \Box\Diamond p \;\oplus \\ & p \to \Diamond p \;\oplus \Diamond\Diamond p \to \Diamond p \;\oplus \Diamond\Box p \to \Box p.\end{aligned}$$

Bezhanishvili [1, 2] introduced a topological semantics for **MIPC**, given by the category **TPSOE** of *perfect augmented Kripke frames* and their morphisms (see 7.0.17 and 7.0.21 below), and proved that **TPSOE** is dually equivalent to the category of *monadic Heyting algebras* and their homomorphisms, which is the class of algebras canonically associated with **MIPC** (see [2]). In this section, we will show that – as it was to be expected – **TPSOE** is isomorphic to the full subcategory **MIPCsp** of **IKsp** whose objects are the **IK**-spaces $\mathcal{G} = \langle X, \leq, R, \mathcal{A}\rangle$ such that R is an equivalence relation.

Definition 7.0.12. (MIPC-space) *An* **MIPC**-space *is an* **IK**-*space* $\mathcal{G} = \langle X, \leq, E, \mathcal{A}\rangle$ *such that* E *is an equivalence relation.*

Definition 7.0.13. (Augmented Kripke frame) *(cf. def 2.1 of [2]) A relational structure* $\langle X, \leq, E\rangle$ *is an* augmented Kripke frame *iff* $\langle X, \leq\rangle$ *is a partial order and* E *is an equivalence relation on* X *such that* $(E \circ \leq) \subseteq (\leq \circ E)$.

Lemma 7.0.14. *The following are equivalent for every relational structure* $\langle X, \leq, E\rangle$*:*

1. $\langle X, \leq, E\rangle$ *is an augmented Kripke frame.*
2. $\langle X, \leq, E\rangle$ *is an* **IK**-*frame such that* E *is an equivalence relation.*

Proof. $(1 \Rightarrow 2)$ Let us show that $(\geq \circ E) \subseteq (E \circ \geq)$: if $x, y, z \in X$ and $x \geq yEz$, then, as E is symmetric, $zEy \leq x$, and so $z \leq vEx$ for some $v \in X$, hence $xEv \geq z$.
$(1 \Rightarrow 2)$ It immediately follows from the definition of **IK**-frame. □

Definition 7.0.15. (Perfect Kripke frame) *(cf. section 3.1 of [2]) A preordered Stone space* $\mathbf{X} = \langle X, \leq, \tau\rangle$ *is a* perfect Kripke frame *iff* $x\uparrow \in K(\mathbf{X})$ *for every* $x \in X$ *and for every clopen subset* U *of* $\mathbf{X}$, $U\downarrow$ *is clopen.*

Proposition 7.0.16. *The following are equivalent for every preordered space* $\mathbf{X} = \langle X, \leq, \tau\rangle$*:*

1. $\mathbf{X}$ *is a quasi Esakia space.*
2. $\mathbf{X}$ *is a Stone space such that for every clopen subset* U, $U\downarrow$ *is clopen.*
3. $\mathbf{X}$ *is a quasi Priestley space such that for every clopen subset* U, $U\downarrow$ *is clopen.*

Proof. See 3.2.7 of [11]. □

From the proposition above it follows that 1) if $\mathbf{X} = \langle X, \leq, \tau\rangle$ is a preordered Stone space such that for every clopen subset U, $U\downarrow$ is clopen, then $\mathbf{X}$ is a Priestley space, hence $x\uparrow \in K(\mathbf{X})$ for every $x \in X$, and so the condition that $\leq$ is point closed in 7.0.15 is redundant, and 2) perfect Kripke frames and quasi-Esakia spaces are one and the same thing.

Definition 7.0.17. (Perfect augmented Kripke frame) *(cf. section 3.1 of [2]) A* perfect augmented Kripke frame *is a structure* $\mathcal{X} = \langle X, \leq, E, \tau\rangle$ *such that*

1. $\langle X, \leq, E\rangle$ *is an augmented Kripke frame.*

2. $\langle X, \leq, \tau\rangle$ *and* $\langle X, (\leq \circ E), \tau\rangle$ *are perfect Kripke frames.*

3. *For every clopen increasing subset* U, $E[U]$ *is clopen.*

Lemma 7.0.18. *For every augmented Kripke frame* $\langle X, \leq, E\rangle$, *and every* $\leq$-*increasing subset* Y, $E[Y]$ *is* $\leq$-*increasing.*

Proof. Let $x \in E[Y]$ and $x \leq z$ and let us show that $z \in E[Y]$: as $x \in E[Y]$ then yEx for some $y \in Y$, so $z \geq xEy$, hence, as $(\geq \circ E) \subseteq (E \circ \geq)$ by 7.0.14, $zEv \geq y$ for some $v \in X$, i.e. $y \leq vEz$, and as Y is increasing and $y \in Y$, then $v \in Y$ and so $z \in E[Y]$. □

Lemma 7.0.19. *(cf. lemma 3.1 (1) of [2]) For every perfect augmented Kripke frame* $\mathcal{X} = \langle X, \leq, E, \tau\rangle$ *and every* $x \in X$, $E[x] = (\leq \circ E)[x] \cap (E \circ \geq)[x]$.

For every perfect augmented Kripke frame $\mathcal{X} = \langle X, \leq, E, \tau\rangle$ let us define $\mathcal{G}_\mathcal{X} = \langle X, \leq, E, \mathcal{A}_\tau\rangle$, where $\mathcal{A}_\tau$ is the **IK**-type algebra of the clopen increasing subsets of $\langle X, \leq, \tau\rangle$, i.e. the modal operations of $\mathcal{A}_\tau$ are $\Box_{(\leq\circ E)}$ and $\Diamond_E$.
For every **IK**-space $\mathcal{G} = \langle X, \leq, E, \mathcal{A}\rangle$ such that E is an equivalence relation let us consider $\mathbf{X}_\mathcal{G} = \langle X, \leq, \tau\rangle$ and define $\mathcal{X}_\mathcal{G} = \langle X, \leq, E, \tau\rangle$.

Proposition 7.0.20.
1. *For every perfect augmented Kripke frame* $\mathcal{X} = \langle X, \leq, E, \tau\rangle$, $\mathcal{G}_\mathcal{X} = \langle X, \leq, E, \mathcal{A}_\tau\rangle$ *is an* **MIPC**-*space.*

2. *For every* **MIPC**-*space* $\mathcal{G} = \langle X, \leq, E, \mathcal{A}\rangle$, $\mathcal{X}_\mathcal{G} = \langle X, \leq, E, \tau\rangle$ *is a perfect augmented Kripke frame.*

Proof. 1. It holds that $\mathbf{X}_{\mathcal{G}_\mathcal{X}} = \langle X, \leq, \tau\rangle$ is a perfect Kripke frame, i.e. a quasi Esakia space, and $\leq$ is a partial order, so $\mathbf{X}_{\mathcal{G}_\mathcal{X}}$ is an Esakia space, and $\mathcal{A}_\tau$ is the algebra of the clopen increasing subsets of $\mathbf{X}_{\mathcal{G}_\mathcal{X}}$, hence D1 holds.
As $\langle X, (\leq \circ E), \tau\rangle$ is a perfect Kripke frame, then for every $x \in X$ $E[x\uparrow] \in K(\mathbf{X}_{\mathcal{G}_\mathcal{X}})$, which is D4, moreover for every $U \in \mathcal{A}_\tau$ (U is a clopen increasing subset of $\mathbf{X}_{\mathcal{G}_\mathcal{X}}$, hence $(X \setminus U)$ is clopen, therefore), $(\leq \circ E)^{-1}[X \setminus U]$ is clopen. It holds that

$$\begin{aligned}
(\leq \circ E)^{-1}[X \setminus U] &= \{z \in X \mid v(\leq \circ E)^{-1}z \text{ for some } v \in (X \setminus U)\} \\
&= \{z \in X \mid z(\leq \circ E)v \text{ for some } v \in (X \setminus U)\} \\
&= \{z \in X \mid E[z\uparrow] \cap (X \setminus U) \neq \emptyset\} \\
&= \{z \in X \mid E[z\uparrow] \not\subseteq U\} \\
&= X \setminus \Box_{(\leq\circ E)}U.
\end{aligned}$$

Hence $\Box_{(\leq\circ E)}U$ is clopen, and it is increasing, for if $z \in \Box_{(\leq\circ E)}U$ and $z \leq y$, then $y\uparrow \subseteq z\uparrow$, and so $E[y\uparrow] \subseteq E[z\uparrow] \subseteq U$, hence $y \in \Box_{(\leq\circ E)}U$. Let us show that for every $U \in \mathcal{A}_\tau$, $\Diamond_E U \in \mathcal{A}_\tau$:

$$\begin{aligned}\Diamond_E U &= \{z \in X \mid E[z] \cap U \neq \emptyset\} \\ &= \{z \in X \mid zEu \text{ for some } u \in U\} \\ &= \{z \in X \mid uEz \text{ for some } u \in U\} \quad (E \text{ is symmetric}) \\ &= E[U].\end{aligned}$$

As $U \in \mathcal{A}_\tau$, then U is a clopen increasing subset of $\mathbf{X}_{\mathcal{G}_\mathcal{X}}$, hence by condition 3 of 7.0.17, $E[U]$ is clopen, and it is increasing, by 7.0.18, which completes the proof of D2. By 7.0.19, for every $x \in X$, $E[x] = (\leq \circ E)[x] \cap (E \circ \geq)[x] = E[x\uparrow] \cap E[x]\downarrow$, which is D5. From D4 and the fact that $\mathbf{X}_{\mathcal{G}_\mathcal{X}}$, being an Esakia space, is a Priestley space, it follows that $E[x] = E[x\uparrow] \cap E[x]\downarrow$ is the intersection of two closed sets, so it is closed, which is D3.

2. By item 3 of 4.1.1 it holds in particular that $(E \circ \leq) \subseteq (\leq \circ E)$, so $\langle X, \leq, E\rangle$ is an augmented Kripke frame. D2 implies that for every clopen increasing subset U, $E[U] = \Diamond_E U$ is clopen. By D1, $\langle X, \leq, \tau\rangle$ is an Esakia space, hence it is a perfect Kripke frame. Let us show that $\langle X, (\leq \circ E), \tau\rangle$ is a perfect Kripke frame: By 7.0.16, it is enough to show that the assignment $x \longmapsto (\leq \circ E)[x]$ defines a continuous map $\rho : \mathbf{X}_\mathcal{G} \longrightarrow \mathbf{K}(\mathbf{X}_\mathcal{G})$. By D4, it holds that $(\leq \circ E)[x] = E[x\uparrow] \in \mathbf{K}^\uparrow(\mathbf{X}_\mathcal{G})$ for every $x \in X$, and as $\mathbf{K}^\uparrow(\mathbf{X}_\mathcal{G})$ is a subspace of $\mathbf{K}(\mathbf{X}_\mathcal{G})$, then it is enough to show that the assignment $x \longmapsto (\leq \circ E)[x]$ defines a continuous map $\rho : \mathbf{X}_\mathcal{G} \longrightarrow \mathbf{K}^\uparrow(\mathbf{X}_\mathcal{G})$. By item 2 of 6.1.5 of [11], $\mathcal{B}^\uparrow_{K^\uparrow(\mathbf{X}_\mathcal{G})} = \{t(U) \cap K^\uparrow(\mathbf{X}_\mathcal{G}) \mid U$ clopen increasing $\} \cup \{m(V) \cap K^\uparrow(\mathbf{X}_\mathcal{G}) \mid V$ clopen decreasing$\}$ is a subbase of $\mathbf{K}^\uparrow(\mathbf{X}_\mathcal{G})$, so it is enough to show that for every clopen increasing subset U of $\mathbf{X}_\mathcal{G}$, $\rho^{-1}[t(U)]$ is clopen. For every clopen increasing subset U of $\mathbf{X}_\mathcal{G}$, $\rho^{-1}[t(U)] = \{x \in X \mid (\leq \circ E)[x] \subseteq U]\} = \Box_{(\leq \circ E)} U$, which is clopen increasing by D2. □

Definition 7.0.21. (Morphism of perfect augmented Kripke frames) *(cf. section 3.1 of [2]) Let $\mathcal{X}_i = \langle X_i, \leq_i, E_i, \tau_i\rangle$ be perfect augmented Kripke frames, $i = 1, 2$. A continuous map $f : \langle X_1, \tau_1\rangle \longrightarrow \langle X_2, \tau_2\rangle$ is a* morphism *iff for every $x, x', y \in X_1$, $z \in X_2$,*

M1. if $x \leq_1 y$ then $f(x) \leq_2 f(y)$.

M2. If $f(x) \leq_2 z$ then $f(x') = z$ for some $x' \in x\uparrow$.

M4'. If $x(\leq_1 \circ E_1)y$ then $f(x)(\leq_2 \circ E_2)f(y)$.

M6'. If $f(x)(\leq_2 \circ E_2)z$ then $z = f(x')$ for some $x' \in (\leq_1 \circ E_1)[x]$.

M5. If $f(x)E_2 z$ then $z \leq_2 f(x')$ for some $x' \in E_1[x]$.

Proposition 7.0.22. *1. For every morphism $f : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ of perfect augmented Kripke frames, f is a p-morphism between the associated* **MIPC***-spaces $\mathcal{G}_{\mathcal{X}_1}$ and $\mathcal{G}_{\mathcal{X}_2}$.*

2. For every p-morphism $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ of **MIPC***-spaces, f is a morphism between the associated perfect augmented Kripke frames $\mathcal{X}_{\mathcal{G}_1}$ and $\mathcal{X}_{\mathcal{G}_2}$.*

Proof. 1. We have to show the conditions M3, M4 and M6 in 3.3.3 hold: M3 is equivalent to the continuity of f, and M6' immediately implies M6. Let us show M4, i.e. assume that xE_1y and show that $f(x)E_2f(y)$: By 7.0.19, it is enough to show that $f(x)(\leq \circ E_2)f(y)$ and $f(x)(E_2 \circ \geq)f(y)$. As $x \leq xE_1y$, then by M4', $f(x)(\leq \circ E_2)f(y)$. As $xE_1y \geq y$, then $x \in (\leq \circ E_1)[y]$ so, by M4', $f(x) \in (\leq \circ E_2)[f(y)]$ i.e. $f(y) \in (\leq \circ E_2)^{-1}[f(x)] = (E_2 \circ \geq)[f(x)]$.
2. We have to show that f is continuous and that M4', M6' in 7.0.21 hold: M3 is equivalent to continuity, and M4' is easily implied by M1 and M4. Let us show M6': assume that $f(x)(\leq_2 \circ E_2)z$, and show that $z = f(x')$ for some $x' \in (\leq_1 \circ E_1)[x]$. By M6, $f(y) \leq_2 z$ for some $y \in (\leq \circ E_1)[x]$, hence, by M2, $z = f(x')$ for some $x' \in y\uparrow$, and as $y \in (\leq \circ E_1)[x]$, then $x' \in (\leq \circ E_1 \circ \leq)[x] = (\leq \circ E_1)[x]$, the last equality being implied by $(E_1 \circ \leq) \subseteq (\leq \circ E_1)$. □

8 Final remarks

Remark 8.0.23. *For every finite linear order $\langle X, \leq\rangle$ with more than one element, the assignment $x \longmapsto (\geq \circ \leq)[x] = x\downarrow\uparrow(= X)$ defines an order-preserving map $\zeta^{\downarrow\uparrow} : \langle X, \leq\rangle \longrightarrow \langle \mathcal{P}(X), \leq^{\uparrow\downarrow}\rangle$ which is not strongly isotone.*

Proof. As $\leq$ is a linear order, then $(\geq\circ\leq) = X \times X$, so $\leq\circ(\geq \circ \leq) \subseteq (\geq \circ \leq)\circ\leq$ and $\geq\circ(\geq \circ \leq) \subseteq (\geq \circ \leq)\circ\geq$, which implies (see 5.1.3 of [11]) that $\zeta^{\downarrow\uparrow}$ is order-preserving. As $\langle X, \leq\rangle$ is a finite linear order, then there exists a maximum element $a \in X$. As $X = \{a\}\downarrow$, then for every $x \in X$, $x\downarrow\uparrow = X \leq^{\uparrow\downarrow} \{a\}$, but since X has more than one element, then $\{a\} \neq X$, so there is no $y \in X$ such that $y\downarrow\uparrow = \{a\}$. □

Remark 8.0.24. *For every finite linear order $\langle X, \leq\rangle$ the assignment $x \longmapsto (\geq\circ \leq)[x] = x\downarrow\uparrow(= X)$ defines a strongly isotone map $\zeta^{\downarrow\uparrow} : \langle X, \leq\rangle \longrightarrow \langle \mathcal{P}_{\geq}(X), \leq^{\downarrow}\rangle = \langle \mathcal{P}_{\geq}(X), \subseteq\rangle$.*

Proof. As $\leq$ is a linear order, then $(\geq\circ\leq) = X \times X$, so $\geq\circ(\geq \circ \leq) \subseteq (\geq \circ \leq)\circ\geq$, which implies (see 5.1.3 of [11]) that $\zeta^{\downarrow\uparrow}$ is order-preserving. For every $x \in X$ and every $F \in \mathcal{P}_{\geq}(X)$, if $x\downarrow\uparrow = X \leq^{\downarrow} F$, then $X = F$, so $x \in X$, $x \leq x$ and $x\downarrow\uparrow = F$. □

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