Sesquilinear forms: A crash-course-survey

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Abstract

It is perhaps a surprising fact that among all sesquilinear forms over a field, only the orthogonal, symplectic and unitary forms are of (essential) interest. A (condensed) survey around this theme will be shown here. No originality is claimed on our investigations as such.

1 Introduction

In this so-called *crash-survey* we will deal with *sesquilinear forms*. It will be shown that there are only a few such ones being of essential interest. We start with some definitions and we will state some well-known "special" cases by means of "Examples-definitions". The symbol F stands for an arbitrary field, ϑ means a field-automorphism of F, and V is a finite dimensional vector space over F.

Definition 1 A map $f : V \times V \to F$ is a *sesquilinear form* (relative to ϑ) if f satisfies

$$\begin{aligned} f(x+y,z) &= f(x,z) + f(y,z), \\ f(x,y+z) &= f(x,y) + f(x,z), \\ f(ax,y) &= a \cdot f(x,y), \\ f(x,ay) &= a^{\vartheta} \cdot f(x,y), \end{aligned}$$

for all $x, y, z \in V$ and $a \in F$.

Examples-definitions Let f be a sesquilinear form.

1. If ϑ is the identity-automorphism of F, then f is a *bilinear* form.

2. a. f is a **symmetric** form, if f is bilinear and $f(x,y) = f(y,x)$ for all $x, y \in V$.

   b. f is a **skew-symmetric** form, if f is bilinear and $f(x,y) = -f(y,x)$ for all $x, y \in V$.

3. If ϑ^2 is the identity-automorphism of F, but ϑ is not, and if also $f(x,y) = (f(y,x))^\vartheta$ for all $x, y \in V$, then f is a **hermitian-symmetric** form.

Let us consider the *radical* Rad(V) of V relative to a sesquilinear form f.

Definition 2 Define Rad(V) as $\mathrm{Rad}(V) := \{v \in V \mid f(x,v) = 0, \forall x \in V\}$. If Rad(V) $= \{0\}$, then f is *non-degenerated* (in short: n-d).

Observe that Rad(V) is a *subspace* of V (i.e. whenever $x, y \in \mathrm{Rad}(V)$ and $a \in F$, it follows that $x \pm y$ and ax (and ay) belong to V.

Examples-definitions Let f be a sesquilinear form.

4. a. Suppose that the characteristic Char(F) of F is not equal to 2. Then f is an **orthogonal** form if f is n-d and symmetric.

   b. Suppose Char(F) = 2. Then f is an **orthogonal** form if f is n-d and symmetric and where it also has to hold that $f(x.x) = 0$ for all $x \in V$.

5. Suppose Char(F) $\neq 2$. Then f is a **symplectic** form if f is n-d and skew-symmetric.

6. f is a **unitary** form. if f is n-d and hermitian-symmetric. Here no restriction relative to Char(F) is required.

Concrete examples

A. Let $x, y \in V$ with $\dim_F(V) = n$. Then, if the coordinates of x ("relative to the standard basis") are $\{\alpha_1, \ldots, \alpha_n\}$, and $\{\beta_1, \ldots, \beta_n\}$ that of y, the form f defined by

$$f(x,y) = \sum_{i=1}^{n} \alpha_i \beta_i$$

gives rise to an **orthogonal** form if Char(F) $\neq 2$.

B. Let here $\dim_F(V) = n = 2m$. Then, for all $x, y \in V$ (as meant in A), the form f defined by

$$f(x,y) = \sum_{i=1}^{m} (\alpha_i \beta_{2m-i+1} - \alpha_{2m-i+1}\beta_i)$$

gives rise to a **symplectic** form if $\mathrm{Char}(F) \neq 2$, or to an **orthogonal** form if $\mathrm{Char}(F) = 2$.

C. for all $x, y \in V$ (as meant in A), the form f defined by

$$f(x,y) = \sum_{i=1}^{n} \alpha_i \beta_i^{\vartheta}$$

gives rise to a **unitary** form if there exists such a field automorphism ϑ of F, being of order 2.

Now observe that all the examples-definitions satisfy the "*commuting-zero-values*"-condition (in short: *czv*):

Whenever $u, t \in V$ satisfy $f(u,t) = 0$, it follows invariably that $f(t,u) = 0$.

The sesquilinear forms f and f' are *equivalent* if there exists a constant $\alpha \in F\backslash\{0\}$ such that $f(x,y) = \alpha f'(x,y)$ for all $x, y \in V$ (i.e. α does not depend on the choice of the elements x, y of V).

We close the Introduction with a Lemma on functionals needed furtheron. Remember that a map $\varphi : V \to F$ is a *linear functional* if $\varphi(ax + by) = a\varphi(x) + b\varphi(y)$ for all $x, y \in V$ and all $a, b \in F$; the map $\psi : V \to F$ is the *zero-functional* if $\psi(v) = 0$ for each $v \in V$. Further it is mentioned that F^* stands for the set $F\backslash\{0\}$.

Lemma 1 *Let $\varphi : V \to F$ and $\psi : V \to F$ be linear functionals with equal zero-sets (i.e., $\varphi(v) = 0$ if and only if $\psi(v) = 0$). Then there exists $\alpha \in F^*$ such that $\varphi(w) = \alpha\psi(w)$ holds independently of the choice of w in V.*

Proof We may assume that none of φ and ψ represent the zero-functional. Therefore, there exists $u \in V$ with $\varphi(u) \neq 0$ **and** $\psi(u) \neq 0$. Consider any $v \in V$ with $\varphi(v) \neq 0$ (whence $\psi(v) \neq 0$). Thus there exist $\alpha \in F^*$ with $\varphi(u) = \alpha\psi(u)$, $\beta \in F^*$ with $\varphi(v) = \beta\psi(v)$, and $\gamma \in F^*$ with $\varphi(u) = \gamma\varphi(v)$.

Now $\varphi(u-\gamma v)=\varphi(u)-\gamma\varphi(v)=0$, whence $0=\psi(u-\gamma v)=\psi(u)-\gamma\psi(v)$. It holds then, that

$$0=\varphi(u)-\gamma\varphi(v)=\alpha\psi(u)-\gamma\beta\psi(v)=\alpha\psi(u)-\beta\psi(u)=(\alpha-\beta)\psi(u).$$

As $\psi(u)\neq 0$, $\alpha=\beta$ follows. The lemma has been proved. $\|$

2 On sesquilinear forms satisfying the *czv* condition

We are going to elucidate the structure of non-trivial sesquilinear forms f satisfying the *czv*-condition. The following subdivision is in order.

(α) Suppose $f(x,x)=0$ for all $x\in V$.

(β) Suppose $f(x,x)\neq 0$ for some $x\in V$, and either

(β.1.a) Let ϑ be of order 2 (i.e. $\vartheta^2=\mathrm{Id}\neq\vartheta$); or

(β.1.b) Let ϑ be of order at least 3 (i.e. $\vartheta^2\neq\mathrm{Id}$); or

(β.2) Let $\vartheta=\mathrm{Id}$.

Re(α) *Suppose $f(x,x)=0$ for all $x\in V$.* Then, for all $u,t\in V$ and $a\in F$, it holds that $f(u+t,\ u+t)=0=f(u,u)+f(u,t)+f(t,u)+f(t,t)=f(u,t)+f(t,u)$, whence also that, for $a\in F^*, af(u,t)=f(au,t)=-f(t,au)=-a^\vartheta f(t,u)=a^\vartheta f(u,t)$. Now f is non-trivial, so $\vartheta=\mathrm{Id}$ follows, implying that f is **skew-symmetric**. Moreover, Rad(V) $\neq$ V as f is not trivial. Thus there exists a subspace W of V for which $f: W\times W\to F$ is **orthogonal** *in case* Char (F) = 2, or **symplectic** *in case* Char F $\neq$ 2.

Re(β) The case (β) gives rise to a plethora of possibilities. If we are *not* in case (β.1.b), then f is certainly *not* symmetric. For there exists $a\in F$ with $a\neq a^\vartheta$; so, as $a\neq 0$ and $f(x,x)\neq 0$ for some $x\in V$, we find for these a and x that $f(ax,x)=af(x,x)\neq a^\vartheta f(x,x)=f(x,ax)$.

Re(β.1.a) *Let ϑ be of order 2 and suppose that $f(x,x) \neq 0$ for some $x \in V$.* Notice that both the maps $\varphi_t : y \mapsto f(y,t)$ and $\psi_t : y \mapsto (f(t,y))^\vartheta$ are **linear functionals**. As f satisfies *czv*, we see that $\varphi_t(v) = 0$ if and only if $\psi_t(v) = 0$. Therefore we are allowed to apply the Lemma. Thus there exists a constant $\alpha_t \in F^*$, *only depending on* $t \in V$, such that

$$f(y,t) = \alpha_t(f(t,y))^\vartheta \quad (\forall y \in V).$$

Analogously, in the same vain, $\alpha_s \in F^*$ exists with

$$f(y,s) = \alpha_s(f(s,y))^\vartheta \quad (\forall y \in V),$$

and $\alpha_{t+s} \in F^*$ exists with

$$f(y,t+s) = \alpha_{t+s}(f(t+s,y))^\vartheta \quad (\forall y \in V).$$

Therefore

$$\begin{aligned} 0 &= f(y,t) + f(y,s) - f(y,t+s) = \\ &= \alpha_t(f(t,y))^\vartheta + \alpha_s(f(s,y))^\vartheta - \alpha_{t+s}(f(t+s,y))^\vartheta = \\ &= (\alpha_t^\vartheta f(t,y))^\vartheta + (\alpha_s^\vartheta f(s,y))^\vartheta - (\alpha_{t+s}^\vartheta f(t+s,y))^\vartheta = \\ &= (f(\alpha_t^\vartheta t + \alpha_s^\vartheta s - \alpha_{t+s}^\vartheta (t+s), y))^\vartheta. \end{aligned}$$

Hence

$$0 = f(\alpha_t^\vartheta t + \alpha_s^\vartheta s - \alpha_{t+s}^\vartheta (t+s), y), \text{ for all } y \in V. \tag{1}$$

Suppose that Rad(V) $= \{0\}$ *and that* $\dim_F(V) \geq 2$.
Then (1) yields $\alpha_t^\vartheta t + \alpha_s^\vartheta s - \alpha_{t+s}^\vartheta (t+s) = 0$. If t and s are *independent* over F, then (1) yields $\alpha_t^\vartheta = \alpha_s^\vartheta = \alpha_{t+s}^\vartheta$, whence that $\alpha_t = \alpha_s = \alpha_{t+s}$.

Next suppose that $0 \neq t = \beta s$ for some $\beta \in F^*$ and let us select $z \in V \backslash \langle t \rangle$. Then, analogously as before, $\alpha_t = \alpha_z$ and $\alpha_s = \alpha_z$ follows. Hence $\alpha_t = \alpha_s$ holds also here!

Therefore we see that α_t does *not* depend on t, i.e. there exists a constant $\alpha \in F^*$ with

$$f(y,t) = \alpha(f(t,y))^\vartheta \text{ for } all\ t, y \in V.$$

So,

$$f(y,t) = \alpha(f(t,y))^\vartheta = \alpha(\alpha(f(y,t))^\vartheta)^\vartheta = \alpha\alpha^\vartheta f(y,t).$$

Now f is non-trivial, and therefore $\alpha\alpha^{\vartheta} = 1$ follows.

If $\alpha = 1$, then f is a **unitary** form. Thus assume $\alpha \neq 1$. Then we will show that there exists $\varepsilon \in F^*$ with $\alpha\varepsilon^{\vartheta} = \varepsilon$. Namely, put $\delta = 1+\alpha$. If $\delta \neq 0$, then

$$\begin{aligned}(1+\alpha)((1+\alpha)^{\vartheta})^{-1} &= (1+\alpha)(1+\alpha^{\vartheta})^{-1} = \\ &= (1+\alpha)(1+\alpha^{-1})^{-1} = \alpha \quad (\text{as } \alpha\alpha^{\vartheta} = 1),\end{aligned}$$

hence indeed $\alpha\varepsilon^{\vartheta} = \varepsilon$ with $\varepsilon = \delta$. If $\delta = 0$, it follows that Char (F) $\neq 2$ (by $\alpha \neq 1$). Hence $\alpha = -1 \neq +1$ and moreover, since ϑ is of order 2, there exists $\beta \in F^*$ with $\beta^{\vartheta} \neq \beta$, thus with $-\beta^{\vartheta} + \beta \neq 0$. Therefore in this case,

$$\alpha = -1 = (-\beta^{\vartheta} + \beta)(-\beta + \beta^{\vartheta})^{-1} = (-\beta^{\vartheta} + \beta)((-\beta^{\vartheta} + \beta)^{\vartheta})^{-1}.$$

Now put $\varepsilon = -\beta^{\vartheta} + \beta$, so that $\alpha\varepsilon^{\vartheta} = \varepsilon$ holds here too. Anyhow, when $\alpha \neq 1$, we find that

$$f(y,t) = \varepsilon(\varepsilon^{\vartheta})^{-1}(f(t,y))^{\vartheta} \text{ for all } t, y \in V,$$

yielding

$$\varepsilon^{\vartheta} f(y,t) = (\varepsilon^{\vartheta} f(t,y))^{\vartheta}.$$

Define the map $\hat{f} : V \times V \to F$ by $\hat{f}(y,t) = \varepsilon^{\vartheta} f(y,t)$ whenever $y, t \in V$. Then the constant $\varepsilon \in F^*$ has the property that for all $t, y \in V$

$$\hat{f}(y,t) = \varepsilon^{\vartheta} f(y,t) = (\varepsilon^{\vartheta} f(t,y))^{\vartheta} = (\hat{f}(t,y))^{\vartheta}.$$

Thus f is *equivalent* to the **unitary** form $\hat{f}$.

Now suppose that Rad(V) $= \{0\}$ *and that* $\dim_F(V) = 1$.
Then, for a fixed $x \in V$ with $f(x,x) \neq 0$ (where $Fx = V$), we see that the map $\hat{f} : V \times V \to F$, defined by

$$\hat{f}(\alpha x, \beta x) := (f(x,x))^{\vartheta} f(\alpha x, \beta x), \text{ with } \alpha, \beta \in F,$$

is a **unitary** form, *equivalent* to f.

Next suppose that Rad(V) $\neq \{0\}$.
The vector space V admits a decomposition as a direct sum of subspaces, by means of $V = W \dot{+} \text{Rad(V)}$, where $f|_W$ being the restricted function $f : W \times W \to F$, is *non-degenerated*, i.e.

$$\{u \in W \mid f(u,w) = 0 \text{ for all } w \in W\} = \{0\}.$$

Therefore, just as it has been argued before, a constant $\lambda \in F^*$ exists, such that $\lambda f|_W$ is a **unitary** form (on W), *equivalent* to $f|_W$. Now define the map $\hat{f} : V \times V \to F$ by means of

$$\hat{f}(w_1 + r_1,\ w_2 + r_2) = \lambda f(w_1, w_2), \text{ for all } w_i \in W,\ r_i \in \mathrm{Rad(V)}.$$

Then $\hat{f}$ is **hermitian symmetric**, and *equivalent* to f.

[Namely :

$$\begin{aligned} & \lambda f(w_1 + r_1,\ w_2 + r_2) = \\ = \ & \lambda f(w_1, w_2) + \lambda f(r_1, w_2) + \lambda f(w_1, r_2) + \lambda f(r_1, r_2) = \\ = \ & \lambda f(w_1, w_2) + \quad 0 \quad + \lambda f(w_1, r_2) + \quad 0 \quad = \\ = \ & \lambda f(w_1, w_2) \ \ \text{(remember: } f(r_2, w_1) = 0 \ \ \textit{implies}\ f(w_1, r_2) = 0) \\ = \ & \hat{f}(w_1 + r_1,\ w_2 + r_2), \end{aligned}$$

and

$$\begin{aligned} & \hat{f}(w_2 + r_2,\ w_1 + r_1) = \lambda f(w_2, w_1) = (\lambda f(w_1, w_2))^\vartheta = \\ = \ & (\hat{f}(w_1 + r_1,\ w_2 + r_2))^\vartheta.] \end{aligned}$$

Re(β.1.b) *Let us suppose that* $\vartheta^2 \neq \mathrm{Id}$ *and assume that* $f(x,x) \neq 0$ *for some* $x \in V$. We spot such an element x. So $x \notin \mathrm{Rad(V)}$. Therefore there exists a subspace W of V containing x, such that $V = W \dot{+} \mathrm{Rad(V)}$. Notice, that $f|_W$ is n-d and that $f|_{\langle x \rangle}$ is n-d. *Suppose* $\dim_F(W) \geq 2$. Consider $T = \{v \in W \mid f(v,x) = 0\}$; it is a subspace of W. If $f(w,x) = 0$ for $w \in W$, then $w \in T$. If $f(w,x) = \alpha \neq 0$, then $w - \bar{\alpha}x \in T$ where $\bar{\alpha} = f(x,x)^{-1}\alpha$, as $f(w - \bar{\alpha}x, x) = f(w,x) - f(\bar{\alpha}x, x) = f(w,x) - \bar{\alpha}f(x,x) = \alpha - \bar{\alpha}f(x,x) = 0$. So $W = \langle x \rangle \perp T$, a perpendicular direct sum decomposition with respect to $f|_W$. Notice also, that now, as $f|_W$ is n-d, also $f|_T$ is n-d (here the *czv*-property of f is used!).
Now, let us suppose that there exists $y \in T$ satisfying $f(y,y) \neq 0$. Observe, there exists $\beta \in F^*$ with $\beta \neq \beta^{\vartheta^2}$. So therefore, as $f(\beta y, \beta y) \neq 0$,

$$\begin{aligned} \frac{(f(\beta y, \beta y))^\vartheta}{f(\beta y, \beta y)} &= \frac{(\beta\beta^\vartheta f(y,y))^\vartheta}{\beta\beta^\vartheta f(y,y)} = \frac{\beta^\vartheta\beta^{\vartheta^2}(f(y,y))^\vartheta}{\beta\beta^\vartheta f(y,y)} = \\ &= \frac{\beta^{\vartheta^2}(f(y,y))^\vartheta}{\beta f(y,y)} \neq \frac{(f(y,y))^\vartheta}{f(y,y)}. \end{aligned}$$

Thus all in all there exists $\bar{y} \in T$ with $f(\bar{y}, \bar{y}) \neq 0$ satisfying

$$(f(\bar{y}, \bar{y}))^\vartheta f(x,x) \neq f(\bar{y}, \bar{y})(f(x,x))^\vartheta.$$

Put $f(\bar{y},\bar{y}) = \alpha \neq 0$ and $f(x,x) = -\gamma \neq 0$; note $\alpha+\gamma \neq 0$. Then, on one hand,

$$\begin{aligned} f(\alpha x + \gamma\bar{y},\, x+\bar{y}) &= f(\alpha x, x) + \gamma f(\bar{y},x) + \alpha f(x,\bar{y}) + \gamma f(\bar{y},\bar{y}) = \\ &= f(\bar{y},\bar{y})f(x,x) + 0 + 0 - f(x,x)f(\bar{y},\bar{y}) = \\ &= 0 \end{aligned}$$

(here, as $\bar{y} \in T, f(\bar{y},x) = 0$, implying $f(x,\bar{y}) = 0$ by the *czv*-condition). On the other hand,

$$\begin{aligned} f(x+\bar{y},\, \alpha x + \gamma\bar{y}) &= \alpha^{\vartheta} f(x,x) + \alpha^{\vartheta} f(\bar{y},x) + \gamma^{\vartheta} f(x,\bar{y}) + \gamma^{\vartheta} f(\bar{y},\bar{y}) = \\ &= \alpha^{\vartheta} f(x,x) + 0 + 0 + \gamma^{\vartheta} f(\bar{y},\bar{y}) \\ &= (f(\bar{y},\bar{y}))^{\vartheta} f(x,x) - (f(x,x))^{\vartheta} f(\bar{y},\bar{y}) \neq 0. \end{aligned}$$

The function f is supposed to satisfy the *czv*-condition, and so we have a contradiction, unless

$$f(y,y) = 0 \text{ for all } y \in T.$$

Now, as $f|_T$ is n-d and non-trivial, we are allowed to apply (α) in this case! That is, $f|_T$ turns out to be skew-symmetric, yielding $\vartheta = \mathrm{Id}$. A contradiction!

Therefore, the supposition "$\dim_F(W) \geq 2$" is false!

So we have here $\dim_F(W) = 1$. Hence the whole drama is essentially played on the line Fx, since $V = \langle x \rangle \perp \mathrm{Rad(V)}$. Thus we see that f is *equivalent* to a so-called **topical** sesquilinear form $\tilde{f}$ for which

$$\tilde{f}(\alpha x, \beta x) = \alpha\beta^{\vartheta} \quad (\alpha, \beta \in F)$$

holds, and where

$$\tilde{f}(v,w) = \tilde{f}(\alpha x, \beta x)$$

if $v = \alpha x + r_1, w = \beta x + r_2$ with $\alpha, \beta \in F$; $r_1, r_2 \in \mathrm{Rad(V)}$.

Finally, we treat case $(\beta.2)$.

Re$(\beta.2)$ *Suppose $f(x,x) \neq 0$ for some $x \in V$, and let $\vartheta = \mathrm{Id}$.* Notice that here both the maps $\varphi_t : y \mapsto f(y,t)$ and $\psi_t : y \mapsto f(t,y)$ are *linear functionals*. It follows from the *czv*-condition that $\varphi_t(v) = 0$ if and only if $\psi_t(v) = 0$. Therefore again we are allowed to apply the Lemma. As in (β.1.a) it follows *that for* $\dim V \geq 2$ *and* Rad(V) $= \{0\}$, a constant $\alpha \in F^*$ exists with $f(y,t) = \alpha f(t,y)$ whenever $t, y \in V$. Choose $x \in V$ as given above, i.e. with $f(x,x) \neq 0$. Therefore, as also $f(x,x) = \alpha f(x,x)$, we see that $\alpha = 1$. That is, f is **orthogonal** in case Char (F) $\neq$ 2 (**symmetric** otherwise). Next *suppose that* $\dim V = 1$ *and* Rad(V) $= \{0\}$. Then $Fx = V$ with $f(x,x) \neq 0$. Hence it follows that

$$f(\alpha x, \beta x) = \alpha\beta f(x,x) = \beta\alpha f(x,x) = f(\beta x, \alpha x),$$

and so f is also **orthogonal** in case Char (F) $\neq 2$ (**symmetric** otherwise).

Next *suppose that* Rad(V) $\neq \{0\}$.Then V=W$\dot{+}$Rad(V), and W is a n-d $f|_W$-space, i.e.,

$$\{u \in E \mid f(u,w) = 0, \ \forall w \in W\} = \{0\}.$$

As $f(x,x) \neq 0$, there exists $w \in W$ with $f(w,w) \neq 0$; namely, put $x = w + r$ with $w \in W$, $r \in$ Rad(V). It follows now from the first part of this rubric (β.2), when applied on $f|_W$ and W, that $f|_W$ is *symmetric.* Now put $v_i = w_i + r_i$ ($w_i \in W$, $r_i \in$ Rad(V)). Then

$$\begin{aligned}
f(v_1, v_2) &= f(w_1 + r_1, \, w_2 + r_2) = \\
&= f(w_1, w_2) + f(w_1, r_2) + f(r_1, w_2) + f(r_1, r_2) = \\
&= f(w_1, w_2) + 0 + 0 + 0 = f(w_1, w_2) = f(w_2, w_1) = \\
&= f(w_2, w_1) + 0 + 0 + 0 = \\
&= f(w_2, w_1) + f(r_2, w_1) + f(w_2, r_1) + f(r_2, r_1) = \\
&= f(w_2 + r_2, \, w_1 + r_1) = f(v_2, v_1).
\end{aligned}$$

Therefore, all in all: f is **symmetric**.

3 The crash-survey

We collect all results in the following **Portemanteau-Result**.

Let f be a non-trivial sesquilinear form over the field F, satisfying the *czv*-property. Then the following holds.

(α) *Suppose* $f(x, x) = 0$ *for all* $x \in V$.
Then ϑ = Id. The form f is **skew-symmetric**. Moreover, if f is n-d, the f is **orthogonal** for Char (F) = 2, and f is **symplectic** for Char (F) $\neq$ 2.

(β) *Suppose* $f(x, x) \neq 0$ *for some* $x \in V$.

(β.1) *Assume* $\vartheta \in$ Aut (F) *is of order* 2.
If Rad(V) = $\{0\}$ and $\dim_F(V) = 1$, then f is equivalent to a **unitary** form.
If Rad(V) = $\{0\}$ *and* $\dim_F(V) \geq 2$, then either f itself is **unitary** form, or there exists a constant $\alpha \in F^*\backslash\{1\}$ satisfying $f(y,t) = \alpha(f(t,y))^{\vartheta}$ for all $t, y \in V$. Observe that $\alpha^{\vartheta} = \alpha^{-1}$. Moreover, there exists a constant $\varepsilon \in F^*$, such that $\hat{f}$, defined by $\hat{f}(y,t) = \varepsilon^{\vartheta} f(y,t)$ for all $y, t \in V$, is a **unitary** form (so f and $\hat{f}$ are equivalent). Namely, if $1 + \alpha \neq 0$, put $\varepsilon = 1 + \alpha$. If $1 + \alpha = 0$, then choose $\beta \in F^*$ with $\beta^{\vartheta} \neq \beta$, and put $\varepsilon = \beta - \beta^{\vartheta}$.
Next suppose Rad(V) $\neq \{0\}$. There exists a non-trivial subspace $W \subseteq V$ with $V = W +$ Rad(V) such that the restricted form $f_W : W \times W \to F$ is equivalent to a *unitary* form $\hat{f}_W$ (this can be accomplished by the method just shown before.) Given $v \in V$, put $v = v_w + v_r$ with $v_w \in W$, $v_r \in$ Rad(V). Then $\hat{f}$, defined by $\hat{f}(\bar{v}, \bar{u}) = \hat{f}_W(\bar{v}_w, \bar{u}_w)$, is a **hermitian symmetric** form equivalent to f.

Now assume $\vartheta \in$ Aut (F) *is of order at least* 3. Then there exists a 1-dimensional subspace $W \subseteq V$ with $V = W \dot{+}$Rad(V). Fix one element w of $W\backslash\{0\}$. Put $v = \alpha_v w + r_v$, for given $v \in W$, with $\alpha_v \in F$, $r_v \in$ Rad(V). Then f is *equivalent* to a so-called **topical** sesquilinear form $\tilde{f}$, defined by $\tilde{f}(v, u) = \alpha_v \alpha_u^{\vartheta}$.

(β.2) *Assume* ϑ = Id.
If Rad(V) = $\{0\}$, then f is **symmetric** (even **orthogonal** if Char (F) $\neq$ 2).
Now suppose Rad(V) $\neq \{0\}$. Then $V = W \dot{+}$Rad(V), W $\neq \{0\}$, where $\{u \in W \mid f(u, w) = 0, \ \forall w \in W\} = \{0\}$. Therefore, it follows from Rad(W) = $\{0\}$ that $f|_W$ is *symmetric*. Given $v \in V$, put $v = v_w + v_r$ with $v_w \in W$, $v_r \in$ Rad(V). Thus f is **symmetric**.

4 References

Among the vast amount of literature concerning the subject of this paper, the following is strongly suggested for further reading.

1. M. Aschbacher, *Finite Group Theory*; Cambridge Un. Pr., Cambridge etc., 1986.

2. E. Artin, *Geometric Algebra*; Wiley Interscience, New York etc., 1957.

3. D.G. Higman (with an appendix by D.E. Taylor), *Classical Groups*; T.U. Eindhoven, Dept. of Math., report 78–WSK–04, August 1978.