# Perp and Star in the Light of Modal Logic

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**Abstract**

This paper is an exploration in the light of modal logic of Dunn's ideas about two treatments of negation in non-classical logics: perp and star. We take negation as an impossibility modal operator and choose the base positive logic to be distributive lattice logic ($DLL$). It turns out that, if we add one De Morgan law and contraposition to $DLL$ (call this system $K_{-}$), then we can prove a natural completeness and hence treat perp in this modal setting. Moreover, star can be dealt with in the extensions of $K_{-}$. Based on these results, a complete table of star and perp semantics for Dunn's kite of negations is given. In the last section, we discuss perp and star in relevance logic and their related logics. The Routley star is interpreted at the end of this paper.

**Keywords** : perp, Routley star, modal logic, relevance logic, Meyer-Routley semantics

## 1 Introduction

In the literature of non-classical logics, two kinds of treatment of negations are among the most eminent:

- $(\neg^*) : x \models \neg A$ iff $x^* \not\models A$;
- $(\neg \perp) : x \models \neg A$ iff $\forall y (y \models A \rightarrow y \perp x)$.

The first one is well-known in relevance logic called Routley star. The second one (called perp[1]) was used in Goldblatt [1974] to model negation in orthologic[2]. It turns out that this agrees with Dunn's gaggle theory about unary operators. Let's see how it fits in gaggle theory.[3] The type for $\neg$ is $t_{\neg} : \vee \mapsto \wedge$. Then the gaggle theoretical definition of $\neg$ is

$$\forall \alpha (\alpha \notin A \vee (\exists y (y \in \alpha \wedge \neg y \in \chi))).^{4}$$

If we define a binary relation $\perp (\alpha, \beta) : \exists y (y \in \alpha \wedge \neg y \in \chi)$, [5] then the above gaggle theoretical definition of negation turns out to be the same as that for Goldblatt's orthonegation:

$$\neg A \in \chi \text{ iff } \forall \alpha (\alpha \in A \Rightarrow \alpha \perp \chi).$$

[1] "Perp" is short for "perpendicular" and means incompatibility in this paper.

[2] A brief history of perp and star can be found in Dunn [1993].

[3] Dunn [1993].

[4] Here we used the so-called U.C.L.A propositions by Anderson-Belnap-Dunn. *A U.C.L.A. proposition* is identified with the set of points where the proposition is true.

[5] This definition is justified by the philosophical interpretation of negations in Dunn [1996]).

These two seeingly-unrelated semantic clauses are closely connected: $\forall xy(x \not\perp y \leftrightarrow y \leq x^*)$. This relation was first discovered and elaborated in Dunn [1993] and Dunn [1996]. If we denote the complement of the perp relation as $C$ (short for compatibility), then we have a more transparent presentation:

$$xCy \Leftrightarrow y \leq x^*.$$

In his dissertation, Vakarelov gave a modal interpretion of negation[6]. Routley et. al [1983] added some axioms about negation to semi-lattice logic and interpreted it as an impossibility operator. Došen [1986] used propositional intuitionistic logic as a test to treat negation as a modal operator. His logic $N$ is obtained by adding one De Morgan law and contraposition to the negation-free fragment of propositional intuitionistic logic. He showed the completeness of $N$ with respect to a class of Kripke frames. Moreover, he discussed various extensions of this logic especially Johansson's minimal logic $J$.

In this paper,we follow a similar line to that in Vakarelov [1977], Routley et al. [1982] and Došen [1986]. We first add to distributive lattice logic a negation as an impossibility operator. The logic $K_-$ for perp turns out to be one half of negative modal logic, a dual logic to Dunn's positive modal logic. The question arises: how can we apply Dunn's translation of perp and star in this background logic? One de Morgan law will play a crucial role in this connection. Let $K_s$ be $K_-$ plus the axiom scheme $\neg(A \wedge B) \vdash \neg A \vee \neg B$ and $\neg\top \vdash \bot$. $K_s$ is the weakest logic that has a star semantics, which is a criteria to determine whether extensions of $K_-$ have a star semantics. The proof for this statement is similar to the proof of completeness of basic modal logic through Jónsson-Tarski Theorem.

Since the Routley star came from relevance logic, it is important to put it back to see its relation to perp. In order to embody the idea that negation and implication are interrelated in relevance logic ($\neg A \equiv_{def} A \rightarrow \bot$), we define the compatibility relation for negation in this way: $xCy \equiv_{def} \exists z(Rxyz)$. The corresponding logic is just as expected: the logic by replacing the background distributive lattice logic in $K_-$ with $B_+$ but the proof is quite different. We use a "way-down" method due to Meyer. The extension with a star semantics can be treated very similarly. At last, we discuss the constant-free fragments of these logics. They can be regarded as a kind of harmonious "marriage" between relevance logic and modal logic.

# 2 Perp and Star in $DLL$

## 2.1 $K_-$ and Perp

In this paper we will consider a standard propositional language $L$. In $L$, we have denumerably many propositional letters $p, r, q, \cdots$; the connectives of $L$ are $\wedge, \vee, \neg$ (and $\rightarrow$). We use capital English letters $A, B, C, \cdots$ or $\alpha, \beta, \cdots$ for formulae of $L$. Capital Greek letters will be used for sets of formulae. The symbols $\forall, \exists, \Rightarrow, \Leftrightarrow, and, or, iff, not$ will be used with the usual meanings in the metalanguage. Define $L^{\top\bot}$ as the language enriched with two constants $\bot$ and $\top$.

For the presentations of logics in this paper, we will use the binary consequence system from Dunn [1995], whose formal objects are pairs of formulas $(A, B)$. They are called *consequence pairs*. $A \vdash_S B$ denotes that $(A, B)$ is derivable in the logic $S$.

**Definition 2.1** *A distributive lattice logic* ($DLL$) is a binary consequence system in the language $L$ containing the following postulates and rules:

- $A \vdash A$

[6] I am thankful to Prof. Dunn for telling me this history of negation.

- $A \vdash B, B \vdash C \Rightarrow A \vdash C$
- $A \wedge B \vdash A, A \wedge B \vdash B$
- $A \vdash B, A \vdash C \Rightarrow A \vdash B \wedge C$
- $A \vdash C, B \vdash C \Rightarrow A \vee B \vdash C$
- $A \vdash A \vee B, B \vdash A \vee B$
- $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

$\lhd$

The logical system $K_-$ is adapted from Dunn's $K_+$. It contains all the axiom schemes of $DLL$,and is closed under the rules in $DLL$ plus the following

- $A \vdash B \Rightarrow \neg B \vdash \neg A$
- $\neg A \wedge \neg B \vdash \neg(A \vee B)$
- $A \vdash \top, \bot \vdash A$
- $\top \vdash \neg\bot$

Note that $K_-$ is defined in the language $L^{\top\bot}$. To put it in an algebraic way, $K_-$ is a distributive lattice with a negative modal operator. So the proof of the completeness has already covered by Dunn's gaggle theory. But, since the theory is too complicated to be presented here while Henkin-style proofs will be used in the other parts of the paper, we choose the conventional completeness proof.

**Definition 2.2** *A compatibility frame* is a triple $\langle W, C, \leq \rangle$ with $W$ is a non-empty set, $\leq$ is a partial order and $C$ a binary relation satisfying the following condition:

$$\text{If } x' \leq x, y' \leq y \text{ and } xCy, \text{ then } x'Cy'.$$

*A perp relation* is the complement of a compatibility relation $C$ and a *perp* frame is a triple $\langle W, T, \leq \rangle$ with $W$ is a non-empty set, $\leq$ is a partial order and $T$ a binary perp relation. In the paper, we will not distinguish between perp frames and compatibility frames although we do between perp relations and compatibility relations.

*A star frame* is also a triple $\langle W, \leq, \star \rangle$ with $W$ a nonempty set, $\leq$ a partial order and $\star$ a function on $W$ satisfying the following condition:

$$\text{If } w \leq v, \text{ then } v^* \leq w^*.$$

$\lhd$

The definitions of truth, validity of consequence pairs and other basic notions are from Dunn [1995].

**Theorem 2.3** *If $A \vdash_{K_-} B$, then $A \models B$.*

**Proof.** Here we just take the characteristic pair $\neg A \wedge \neg B \vdash \neg(A \vee B)$ as an example. $w \models \neg A \wedge \neg B \Rightarrow \forall v(wCv \rightarrow v \not\models A \vee B) \Rightarrow w \models \neg(A \vee B)$.

QED

Now we go for the more difficult completeness. Compared to the proof of completeness in Dunn [1995], our proof here is easier because we don't need to coordinate two accessibility relations $R_\Box$ and $R_\Diamond$. In the following we will require that prime theories $P$ satisfy additionally: $\top \in P$ and $\bot \notin P$. The following lemma is an analogy of Lindenbaum Lemma in Dunn [1995].

**Lemma 2.4** *(Pair Extension Lemma 1) Let $A$ be a consistent theory and $B$ be a set of formula closed under disjunction. If $A \cap B = \emptyset$, then there is a prime theory $P$ such that $A \subseteq P$ and $P \cap B = \emptyset$.*

Now we define the canonical model $M_c := \langle W_c, C_c, \subseteq, V_c \rangle$. Its universe consists of all prime theories of $K_-$. $\leq_c$ is the set inclusion. Define $PC_cQ$ iff, for all formula $\varphi$, if $\neg\varphi \in P$, then $\varphi \notin Q$. And, for all propositional letters $p$, $P \models_c p \equiv_{def} P \in v_c(p) \equiv_{def} p \in P$.

**Lemma 2.5** *In the above defined canonical frame, $\neg\varphi \in P$ iff $\forall Q(PC_cQ \rightarrow \varphi \notin Q)$.*

**Proof.** The direction from left to right is trivial. For the other direction, we show by contraposition. Suppose that $\neg\varphi \notin P$. Now we need to show that there is a prime filter $Q$ such that $PC_cQ$ and $\varphi \in Q$. First note that $\varphi$ is not equivalent to $\bot$ for otherwise $\top \notin P$, which is agaist our definition of prime theories. Therefore $[\varphi)$ is a consistent theory where $[\varphi)$ is a theory generated by $\varphi$. Moreover, $[\varphi) \cap \neg^{-1}(P) = \emptyset$. For, otherwise, there is $\psi$ such that $\neg\psi \in P$ and $\varphi \vdash \psi$, which implies that $\neg\varphi \in P$. It is easy to check that $\neg^{-1}(P)$ is closed under disjunction.[7] By appealing to Pair Extension Lemma, we have that there is a prime theory $Q$ such that $\neg^{-1}(P) \cap Q = \emptyset$ and $[\varphi) \subseteq Q$. So we are done.

QED

**Lemma 2.6** *(Truth Lemma) $P \models \varphi$ iff $\varphi \in P$.*

**Proof.** The nontrivial is to show the case for $\varphi := \neg\psi$ for some $\psi$. Assume that $\neg\psi \in P$. According to the above lemma, we have $\forall Q(PC_cQ \Rightarrow \psi \notin Q$. By I.H. $\forall Q(PC_cQ \rightarrow Q \not\models \psi)$. That is to say, $P \models \neg\psi$. This argument can be reversed to show the other direction.

QED

**Theorem 2.7** *(Completeness) If $A \models B$, then $A \vdash B$.*

**Proof.** The proof is again an application of Pair Extension Theorem as well as the above lemma. We may assume that $A$ is not equivalent to $\bot$. Suppose that $A \nvdash B$. Take $\Delta$ to be the disjunctive closure of $B$. And $[A)$ is a consistent theory and $[A) \cap \Delta = \emptyset$. By P.E.T, there is a prime theory $Q$ such that $[A) \subseteq Q$ and $B \notin Q$. That is to say, $A \not\models_{M_c} B$.

QED

This theorem says that, if we pick $DLL$ as the background logic, then $K_-$ is the weakest logic that has a perp semantics. Next we will discuss the extensions of this logic especially $K_S$ with which star matches, where $K_s$ is the logic $K_-$ plus $\neg(A \wedge B) \vdash \neg A \vee \neg B$ and $\neg\top \vdash \bot$.

## 2.2 Representation of Ockham Lattices

**Definition 2.8** *An Ockham lattice* is a structure $(D, \wedge, \vee, \neg, 0, 1)$, where $(D, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is a dual endomorphism:

- $\neg(x \vee y) = \neg x \wedge \neg y$;
- $\neg(x \wedge y) = \neg x \vee \neg y$.

$\triangleleft$

[7] This is the very place where $\neg A \wedge \neg B \vdash \neg(A \vee B)$ comes in.

Dunn once raised a question: "it would be nice to somehow fit the Ockham lattices into the framework of the paper (on perp and star) $\cdots$ and represent them." In this section,we will give a set theoretic representation of Ockham lattices, which is similar to that of De Morgan lattices. In the next section, we will use this representation to prove the completeness of $K_S$ with respect to the class of star frames.

Observation: Let $D$ be an Ockham lattice, $a, b \in D$. If $a \leq b$, then $\neg b \leq \neg a$.

**Definition 2.9** Let $\mathcal{F} = \langle W, \star \rangle$ be a star frame. The *full concrete lattice* $\mathcal{F}^+$ associated with $\mathcal{F}$ is the power set [8] of $W$ with a unary operator $\neg$: $\neg(X) = W - \{w \in W : w^* \in X\}$. Any sublattice of a full concrete lattice is called a *concrete lattice.* It is easy to check that $(\mathcal{F}^+, \cap, \cup, \neg)$ is an Ockham lattice (see below). Conversely, for any Ockham lattice $L$, we can define a star frame $L_+ = \langle W, \star \rangle$ associated with $L$ where $W$ is the set of all prime filters on $L$ and $P^* := \{a : \neg a \notin P\}$. Finally, we call $(L_+)^+$ the *embedding lattice* of $L$ (denoted as: $Em(L)$).

$\lhd$

Since the contexts are always clear, we will not use differnt notations to distinguish the $\neg$ in the abstract Ockham lattice and that in the concrete Ockham lattice, which is also true to the following $\wedge$ and $\vee$. Now we will just show that, for any Ockham lattice $L$, it can be embedded into its embedding lattice, which is an analogy of Jónsson-Tarski Theorem.

**Theorem 2.10** *Any Ockham lattice L is isomorphic to a sublattice of its embedding lattice Em(L).*

**Proof.** First we show that $Em(L)$ is an Ockham lattice.

1. $\neg(X \cup Y) = \neg(X) \cup \neg(Y)$: $P \in \neg(X \cup Y) \Leftrightarrow P^* \notin X \cup Y \Leftrightarrow P^* \notin X, P^* \notin Y \Leftrightarrow P \in \neg(X) \wedge P \in \neg(Y)$.
2. $\neg(X \cap Y) = \neg(X) \cup \neg(Y)$ can be shown similarly.

Probably the crucial part is to show that $\star$ maps prime filters to prime filters, or $L_+$ is well-defined. In fact, the conditions in the definition of Ockham lattices are the *minimum* requirements that make the above defined $\star$ mapping prime filters to prime filters as we can see below.

1. Since $\neg 1 \leq 0$ and $0 \notin P$, $1 \in P^*$;
2. Since $1 \leq \neg 0$ and $1 \in P$, $\neg 0 \in P$ and hence $0 \notin P^*$;
3. Let $a \leq b$ and $a \in P^*$. Then $\neg b \leq \neg a$ and $\neg a \notin P$. It follows that $\neg b \notin P$. That is to say, $b \in P^*$;
4. Let $a, b \in P^*$. It follows that $\neg a \notin P$ and $\neg b \notin P$. Since $P$ is prime, $\neg a \vee \neg b \notin P$. According to De Morgan laws, $\neg(a \wedge b) \notin P$. So, $a \wedge b \in P^*$.
5. Let $a \vee b \in P^*$. That is to say, $\neg(a \vee b) \notin P$. Alternatively, $\neg a \wedge \neg b \notin P$. $\neg a \notin P$ or $\neg b \notin P$. So, $a \in P^*$ or $b \in P^*$.

[8] In fact, we can make an intuitionistic case by defining $\mathcal{F}$ as the set of all increasing subset of $W$ ($X$ is increasing iff $a \leq b$ and $a \in X$ implies $b \in X$). And all the notions below would still be well-defined, all the propositions below would still hold by similar proofs.

Define $f : a \rightarrowtail \{P : a \in P\}$. It is easy to see that it is one-to-one from $L$ to $Em(L)$ because of Prime Filter Separation property. Now we need to show that it preserves the operators. Here we only show the case for the non-trivial operator $\neg$.

$$P \in f(\neg a) \Leftrightarrow \neg a \in P \Leftrightarrow a \notin P^* \Leftrightarrow P^* \notin f(a) \Leftrightarrow P \in \neg(f(a)).$$

QED

As we can easily see, this is a generalization of Bialynicki-Birula and Rasiowa's representation of De Morgan lattices[9], for De Morgan lattices are always Ockham lattices. The representation theorem here is the most crucial step in the proof of completeness of $K_s$ with respect to the class of star frames.

## 2.3 Algebraic Characterization of Star Frames

In the following we will show that Ockham lattices to $K_s$ is the same as Boolean algebras with operators to modal logic, or more precisely, as De Morgan lattices to $R_{fde}$. We will omit some regular proofs. A convention: we can take a formula $\varphi$ as a consequence pair $\top \vdash \varphi$. Let $N$ be a deductive system. Then we set $N \vdash \varphi \equiv_{def} \top \vdash_N \varphi$ and $F \models \varphi \equiv_{def} \top \models_F \varphi$. Some other notations are clear from contexts.

**Lemma 2.11** *The following two propositions are about the consequence pair* $\neg\top \vdash \bot$*:*

- *Let* $F$ *be a perp frame. Then* $\neg\top \models_F \bot$ *iff* $F \models \forall x \exists y(xCy)$.
- *The consequence pair* $\neg\top \vdash \bot$ *is canonical in the following sense: If* $K'$ *be any consistent extension of* $K_-$ *and* $\neg\top \vdash_{K'} \bot$*, then the canonical frame of* $K'$ *satisfies* $\forall x \exists y(xCy)$.

**Proof.** Part (1) is trivial. For the second part, we will apply the Pair Extension Theorem. It suffices to show that, in the canonical frame of $N'$, for any prime theory $P$, there is a prime theory $Q$, such that $\neg^{-1}(P) \cap Q = \emptyset$. First note that $\neg^{-1}(P)$ is closed under disjunction and $D := \{\top\}$ can be regarded as a consistent theory. It is easy to check that $\neg^{-1}(P) \cap D = \emptyset$ for otherwise, since $\neg\top \vdash \bot$, $\bot \in P$, a contradiction. Now the lemma follows immediately from the Pair Extension Lemma.

QED

**Theorem 2.12** $A \vdash_{K_s} B$ *iff* $A \vdash B$ *is valid on all the frames satisfying;* $\forall x \exists y \forall z(xCz \leftrightarrow z \sqsubseteq y)$*. In particular,* $\vdash_{K_s} B$ *iff* $B$ *is valid on all the frames satisfying;* $\forall x \exists y \forall z(xCz \leftrightarrow z \sqsubseteq y)$.

**Proof.** It follows from the above lemma, Theorem 2.7 here, Theorem 2.2 and Lemma 2.3 in Restall [2000].

QED

Let $\mathcal{F}$ be the class of star-crossed perp frames, or frames satisfying the above frame condition. As we can see from the above theorem, the logic $K_s$ is $\Lambda_F$.

Above we take perp as primitive and search the necessary and sufficient conditions for perp to be star-crossed by regarding star as secondary. Now we will reverse this direction by taking star as primitive. Of course, we can define perp from star by Dunn's famous translation: $xCy :\equiv y \leq x^*$. We will show a completeness theorem by algebraic method.

[9] Dunn [1986].

It is instructive to keep in mind the slogan in algebraic logic: formulas as terms denoting propsitions.[10]

**Definition 2.13** Let $\Phi$ be a set of variables, $L$ be an Ockham lattice. An assignment for $\Phi$ is a function $\theta : \Phi \to L$, We can extend uniquely $\theta$ to a meaning function $\bar{\theta}$: $Form(\Phi) \to L$ satisfying:

- $\bar{\theta}(\bot) := 0$;
- $\bar{\theta}(p) := \theta(p)$;
- $\bar{\theta}(\neg s) := \neg\bar{\theta}(s)$;
- $\bar{\theta}(s \vee t) := \bar{\theta}(s) \vee \bar{\theta}(t)$.

$\lhd$

**Theorem 2.14** *Let $\varphi$ be a formula, $\mathcal{F}$ be a star frame, $\theta$ an assignment and $w \in F$. Then*

- $(\mathcal{F}, \theta), w \models \varphi$ *iff* $w \in \bar{\theta}(\varphi)$;
- $\mathcal{F} \models \varphi$ *iff* $\mathcal{F}^+ \models \varphi \approx \top$;
- $\mathcal{F} \models \varphi \leftrightarrow \psi$ *iff* $\mathcal{F}^+ \models \varphi \approx \psi$;

**Proof.** Here we only note that $\bar{\theta}(\neg\varphi) = \neg\bar{\theta}(\varphi)$ where the second $\neg$ is the unary operator on $\mathcal{F}^+$.

QED

**Corollary 2.15** *Let $\mathcal{K}$ be a class of star frames and $\varphi$ is a formula. Then,*

- $\mathcal{K} \models \varphi$ *iff* $Cm\mathcal{K} \models \varphi \approx \top$;
- $\mathcal{K} \models \varphi \leftrightarrow \psi$ *iff* $Cm\mathcal{K} \models \varphi \approx \psi$;

*where* $Cm(\mathcal{K}) := \{F^+ : F \in \mathcal{K}\}$.

Note that $Cm\mathcal{K}$ is a class of Ockham lattice,

**Lemma 2.16** *The Lindenbaum-Tarski algebra of $K_s$ is an Ockham lattice. The* Lindenbaum-Tarski Algebra *of $K_s$ is the structure:*

$$\mathcal{L}_{K_s}(\Phi) := (Form(\Phi)/\equiv, \vee, \wedge, \neg)$$

*where the definition of these operators are evident as usual.*

**Theorem 2.17** $\varphi \vdash_{K_s} \psi$ *iff* $\mathcal{L}_{K_s}(\Phi) \models \varphi \wedge \psi \approx \varphi$. *In particular,* $\vdash_{K_s} \varphi$ *iff* $\mathcal{L}_{N_s}(\Phi) \models \varphi \approx \top$.

**Proof.** The left-to-right is obvious by just taking the canonical valuation. In order to show the other direction,we will just use the connection between semantic assignment and syntactic substitution.

QED

**Theorem 2.18** *(Ockham Lattices Algebraize $K_s$)* $\varphi \vdash_{K_s} \psi$ *iff* $\mathcal{OL} \models \varphi \vee \psi \approx \psi$ *where $\mathcal{OL}$ is the class of Ockham lattices. In particular,* $\vdash_{K_s} \psi$ *iff* $\mathcal{OL} \models \top \approx \psi$

[10] Dunn et al.[2001] or Blackburn et al.[2001].

**Proof.** The proof of the right-to-left direction is just a combination of the above two propositions. For the other direction,it is just an induction on the length of the proof of $\varphi \vdash \psi$.

QED

This is of course a very interesting result. But, to logicians, completeness is much more important. In the following, we will concentrate on that. As expected, the crucial step is the above representation in Section 2.2 which connect the abstract Ockham lattices to concrete Ockham lattices. To put it in a professional way, it connects the validity to provability.

**Lemma 2.19** *(Soundness) Let $\mathcal{F}$ be a star frame. If $\varphi \vdash_{K_s} \psi$, then $\varphi \models_F \psi$. And hence, $Cm(\mathcal{K}_s) \models \varphi \vee \psi \approx \psi$ where $\mathcal{K}_s$ is the class of star frames while $K_s$ is a deductive system. In particular, if $\vdash_{K_s} \psi$, then $\models_F \psi$. And hence, $Cm(\mathcal{K}_s) \models \top \approx \psi$*

**Proof.** We check step by step.

- contraposition is preserved in $\mathcal{F}$. Let $\mathcal{F} \models \varphi \vdash \psi$ and $w \models \neg\psi$. It follows that $w^* \not\models \psi$. By the assumption, we have $w^* \not\models \varphi$.So $w \models \neg\varphi$.
- It is easy to see from the above proof that $\mathcal{F} \models \neg\varphi \vee \neg\psi \vdash \neg(\psi \wedge \varphi)$ and $\mathcal{F} \models \neg(\varphi \vee \psi) \vdash \neg\psi \wedge \neg\varphi$.
- $\mathcal{F} \models \neg(\varphi \wedge \psi) \vdash \neg\varphi \vee \neg\psi$. $w \models \neg(\varphi \wedge \psi) \Rightarrow w^* \not\models \varphi \wedge \psi \Rightarrow w^* \not\models \varphi$ or $w^* \not\models \psi \Rightarrow w \models \neg\varphi \vee \neg\psi$.
- $\mathcal{F} \models \neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$. The proof is similar.

The proof of the second part is just application of the Corollary 4.5.

QED

**Theorem 2.20** *(Completeness of $K_s$ with respect to $\mathcal{K}_s$) If $\mathcal{K}_s \models \varphi \vdash \psi$, then $\varphi \vdash_{K_s} \psi$.*

**Proof.** We prove by contraposition. Suppose that $\varphi \nvdash_{K_s} \psi$. Then by Theorem 4.8, there is an Ockham lattice $L$ such that $L \not\models \varphi \approx \varphi \wedge \psi$. It follows from representation that there is a complex lattice $L_c$ such that $L_c \not\models \varphi \approx \varphi \wedge \psi$. Since any variety is closed under subalgebra, there is a full complex lattice $L_0$ such that $L_0 \not\models \varphi \approx \varphi \wedge \psi$. Let $F$ be the underlaying star frame for $L_0$. By Theorem 4.4, $\varphi \not\models_F \psi$. So, if $\mathcal{K}_s \models \varphi \vdash \psi$, then $\varphi \vdash_{K_S} \psi$.

QED

This theorem says that $K_s$ is the weakest normal modal logic with a star semantics. Indeed, $K_s$ to star semantics is the same as $K$ to Kripke semantics and as $B^+$ to Boolean Meyer-Routley semantics.

## 2.4 Bidirectional Frames and Galois Connected Negations

In his several papers, Dunn talked about Galois connected negations: the negations satisfies the following Galois property:

$$A \vdash [\neg]B \text{ iff } B \vdash [\sim]A.$$

He also mentioned that the relations of the perp frames for these two negations: $\perp^{\neg}$ and $\perp^{\sim}$ are just converse to each other. In this section, we just formalize his argument and show that a logic system with Galois connected negations is complete with respect to *the* class of all bidirectional compatibility frames. Since it is well-known that bidirectional frames are appropriate to tense logics, the completeness below convinces us that the Galois-connective logic is the *minimal negative* tense logic in disguise. [11]

Here we will talk about two negative modalities, or more precisely, two impossibility modalities $[\neg]$ and $[\sim]$.[12] Their intended interpretations are as follows:

- $x \models [\neg]\varphi$ iff $\forall y(xR_{\neg}y \to y \not\models \varphi)$
- $x \models [\sim]\varphi$ iff $\forall y(xR_{\sim}y \to y \not\models \varphi)$

Why did we use the above *compatibility* frames to interpret them? The main reason is the following observation:

**Lemma 2.21** [13]*If* $A \vdash [\neg]B \Leftrightarrow B \vdash [\sim]A$, *then* $[\neg]A \wedge [\neg]B \vdash [\neg](A \vee B)$ *and* $[\sim]A \wedge [\sim]B \vdash [\sim](A \vee B)$

These are exactly two defining axiom schema for the two impossibility modalities, respectively! The following deductive system is just a combination of the two systems $K_{\neg}$ and $K_{\sim}$ with one defining rule:

$$A \vdash [\neg]B \text{ iff } B \vdash [\sim]A$$

**Lemma 2.22** [14]*The following two are equivalent:*

- $A \vdash [\neg]B$ *iff* $B \vdash [\sim]A$
- *the conjunction of the following principles:*
  1. $A \vdash B \Rightarrow [\neg]B \vdash [\neg]A$,
  2. $A \vdash B \Rightarrow [\sim]B \vdash [\sim]A$
  3. $A \vdash [\neg][\sim]A$
  4. $A \vdash [\sim][\neg]A$

Although we can give a more concise presentation of the expected system if we add the first, the addition of the second will give a much more transparent presentation. Let $T_{-}$ denote the deductive system $DLL$ plus the follong principles:

- $A \vdash \top, \bot \vdash A$
- $\top \vdash [\neg]\bot$, $\top \vdash [\sim]\bot$.
- $A \vdash B \Rightarrow [\neg]B \vdash [\neg]A$,
- $A \vdash B \Rightarrow [\sim]B \vdash [\sim]A$
- $A \vdash [\neg][\sim]A$
- $A \vdash [\sim][\neg]A$

As Lemma 5.2 shows, we can replace the last four by a single rule:

$$\text{(Galois Connection) } A \vdash [\neg]B \text{ iff } B \vdash [\sim]A$$

[11]The minimal tense logic is called $K_t$.

[12]The reason we use $[\neg]$ and $[\sim]$ instead of $\neg$ and $\sim$ because we want to leave some rooms for their duals $\langle\neg\rangle$ and $\langle\sim\rangle$, which are included in Dunn's "Chinese Restaurant Menu".

[13]Dunn [1996].

[14]Dunn [1996].

**Definition 2.23** (Bidirectional frames) A bidirectional frame is a a triple $\langle W, R_\neg, R_\sim\rangle$ where W is a nonempty set, $R_\neg$, and $R_\sim$ are two binary relations and $R_\neg^{-1} = R_\sim$. A *bidirectional compatibility frame* is a 4-tuple $\langle W, R_\neg, R_\sim, \leq\rangle$ where both $\langle W, R_\neg, \leq\rangle$ and $\langle W, R_\sim, \leq\rangle$ are compatibility frames and $R_\neg^{-1} = R_\sim$. And all the formulas are modelled as expected (especially for the formulas with the two impossibility modalities).

$\lhd$

**Theorem 2.24** *(Soundness) Let $F$ be a bidirectional (compatibility) frame. If $A \vdash_{T_-} B$, then $A \models_F B$.*

**Proof.** Here we just note that the two axiom schema $A \vdash [\neg][\sim]A$ and $A \vdash [\sim][\neg]A$ forces the two accessibility relations $R_\neg$ and $R_\sim$ are converse to each other.

QED

Now we go for the completeness. Our definition of canonical frame for $T_-$ is just a combination of the two canonical frames for $K_\neg$ and $K_\sim$. Here are the two canonical relations:

- $PR^c_\neg Q$ iff $([\neg]\varphi \in P \Rightarrow \varphi \notin Q)$
- $PR^c_\sim Q$ iff $([\sim]\varphi \in P \Rightarrow \varphi \notin Q)$

According the canonical model theorem in section 2.1, the only thing that we need to show is that the canonical frame is bidirectional.

**Theorem 2.25** *If $PR^c_\neg Q$, then $QR^c_\sim P$ and vice versa.*

**Proof.** Let $PR^c_\neg Q$ and $[\sim]\varphi \in Q$. We need to show that $\varphi \notin P$. Suppose that $\varphi \in P$. Since $\varphi \vdash_{T_-} [\neg][\sim]\varphi$, $[\neg][\sim]\varphi \in P$. By the assumption that $PR^c_\neg Q$, $[\sim]\varphi \in Q$, which contradicts our assumption that $[\sim]\varphi \in Q$.

QED

**Theorem 2.26** *$A \vdash_{T_-} B$ iff $A \models_F B$ for all bidirectional compatibility frames $F$.*

This theorem says that the class of bidirectional compatibility frames to Galois connected negations is the same as that of compatibility frames to pre-minimal negation or as that of star frames to Ockham negation.

Question: Can we enrich our language to include $\langle\neg\rangle$ and $\langle\sim\rangle$ in our above enterprize and achieve a natural frame completeness? Their intended interpretations are as follows:

- $x \models \langle\neg\rangle A$ iff $\exists y(xR_\neg y \wedge y \not\models A)$
- $x \models \langle\sim\rangle A$ iff $\exists y(xR_\sim y \wedge y \not\models A)$

First we look at possible classical semantics. Just like in Positive Modal Logic,[15] $A \vdash [\neg][\sim]A$ and $A \vdash [\sim][\neg]A$ are not enough to deal with the enriched language with two additional modalities $\langle\neg\rangle$ and $\langle\sim\rangle$. If we want to make the logic frame complete, we have to add two more dual to the mentioned: $\langle\neg\rangle\langle\sim\rangle A \vdash A$ and $\langle\sim\rangle\langle\neg\rangle A \vdash A$. If we include these two formulas in the deductive system, it is easy to show the completeness just as in Dunn [1995].

So the hope lies in the so-called new semantics where we consider increasing valuation. But another difficulty appears. The Galois property cannot force the relations $R_\neg$ and $R_\sim$ to be converse to each other.

[15]Dunn [1995] and Celani and Jansana [1997].

**Lemma 2.27** *$A \vdash [\neg][\sim]A$ corresponds to $R_\neg \subseteq (R_\sim \cdot \leq^{-1})^{-1}$; $A \vdash [\sim][\neg]A$ corresponds to $R_\sim \subseteq (R_\neg \cdot \leq^{-1})^{-1}$.*

If we can strictly condense the frames, then we have $R_\neg = R_\sim^{-1}$ since $R_\neg = R_\neg \cdot \leq^{-1}$ and $R_\sim = R_\sim \cdot \leq^{-1}$. But it seems that we can not strictly condense the relations on canonical models.[16] Therefore the natural and successful way to include the above two negative modalities is to add to $T_-$(the first four are subminimal while the last two are characteristic)

- $A \vdash B \Rightarrow \langle\neg\rangle B \vdash \langle\neg\rangle$,
- $A \vdash B \Rightarrow \langle\sim\rangle B \vdash \langle\sim\rangle A$,
- $\langle\neg\rangle\bot \vdash \top$,
- $\langle\sim\rangle\bot \vdash \top$
- $\langle\neg\rangle\langle\sim\rangle A \vdash A$ and
- $\langle\sim\rangle\langle\neg\rangle A \vdash A$.

We call this new system $T_-^4$. It is easy to show either by Dunn's classical method or Celani and Jansana's "new" method the following theorem:

**Theorem 2.28** *In classical semantics, $A \vdash_{T_-^4} B$ iff $A \models_F B$ for all bidirectional frames $F$; in intuitionistic semantics, $A \vdash_{T_-^4} B$ iff $A \models_F B$ for all bidirectional* compatibility *frames $F$.*

## 2.5 Dunn's Kite of Negations

**Definition 2.29** Let's call the negation in $K_-$ *preminimal negation*. Other types of negations can be defined by adding new principles:

- *Quasi-Minimal negation*: Preminimal negation + $(A \vdash \neg\neg A)$;
- *Intuitionistic negation*: Quasi-minimal negation + $(A \wedge \neg A \vdash \bot)$;
- *De Morgan negation*: Quasi-minimal negation + $(\neg\neg A \vdash A)$;
- *Ortho-negation*: Intuitionistic negation + De Morgan negation.

$\lhd$

We will treat one by one. It is easy to see that neither quasi-minimal nor intuitionistic negation has a star semantics. As we have remarked, $K_s$ is the weakest logic with a star semantics (with $DLL$ as background logic) but $\neg(A \wedge B) \vdash \neg A \vee \neg B$ does not hold in either of them.

**Theorem 2.30** *$A \vdash \neg\neg A$ corresponds to $\forall x \forall y(xCy \to yCx)$; moreover, it is canonical.*

**Proof.** The first part is obvious. Now we just show Part (2). Assume that $PC_cQ$ and $\neg\varphi \in Q$. We need to show that $\varphi \notin P$. Suppose that $\varphi \in P$. Since $\varphi \vdash \neg\neg\varphi$, $\neg\neg\varphi \in P$. Therefore, $\neg\varphi \notin Q$, which contradicts our assumption that $\neg\varphi \in Q$.

QED

So, for quasi-minimal negation, the perp frame condition is that $C$ is symmetric.

**Theorem 2.31** *$\neg A \wedge A \vdash \bot$ corresponds to $\forall x(xCx)$. And it is canonical.*

[16] It seems that the relations $R_s$ in Celani and Jansana [1997] can not be condensed, which is also true here.

**Proof.** We only show Part (2). Assume that $\neg\varphi \in P$. Since $\varphi \wedge \neg\varphi \vdash \bot$, $\varphi \notin P$. That is to say, $PC_cP$.

QED

So, the perp frame condition for intuitionistic negation is that $C$ is both symmetric and reflexive.

It is well-known that de-Morgan negation has a very natural star semantics. Let $D$ denote $K_- + (A \vdash \neg\neg A) + (\neg\neg A \vdash A)$. Then it is easy to show that $\neg(A \wedge B) \vdash_D \neg A \vee \neg B$ and $\neg\top \vdash_D \bot$. By Theorem 2.20, the star semantics is guaranteed.

**Theorem 2.32** *$\neg\neg A \vdash A$ corresponds to the star-frame condition: $\forall w\ (w^{**} \leq w)$. And it is canonical.*

**Proof.** The proof of the correspondence part with star is similar to "play with" the binary perp relation. Now we show the second part. Assume that $\varphi \in P^{**}$. It follows that $\neg\varphi \notin P^*$. Further, we have $\neg\neg\varphi \in P$. Since $\neg\neg\varphi \vdash \varphi$, $\varphi \in P$.

QED

As for the perp frame conditions for de Morgan negations, we cite the results from Restall [2000]:

**Theorem 2.33** *Let $F$ be a perp frame. $\neg\neg A \models_F A$ iff $F \models \forall x \exists y (xCy \wedge \forall z(yCz \to z \leq x))$. And $A \vdash_D B$ iff it is valid on all the perp frames which are symmetric and additionally satisfy the condition: $\forall x \exists y (xCy \wedge \forall z(yCz \to z \leq x))$.*

We can say little more about the perp semantics for ortho-negation because it is a combination of intuitionistic and de Morgan negations and the defining consequence pairs are canonical. Also the star semantics for orthonegation is obvious.

**Theorem 2.34**
- *$A \wedge \neg A \vdash \bot$ corresponds to $\forall w(w \leq w^*)$; $\neg\neg A \vdash A$ corresponds to $\forall(w^{**} \leq w)$.*
- *Both consequence pairs are canonical with respect to star semantics.*
- *Combining the above two conditions together, we have $A \vdash_{K_o} B$ iff it is valid on all star frames satisfying that $w^* = w$ where $K_o$ is $K_-$+ Ortho-negation axioms.*

Since we choose the background lattice logic to be distributive, ortho-negation collapses to classical negation. For $x \models \neg A$ iff $x^* \not\models A$ iff $x \not\models A$. Now we summarize what we have achieved for the semantical analysis of Dunn's kite of negations. Let $(\star)$ and $(*)$ be the frame conditions that $(\leq \circ R \circ \leq^{-1}) \subseteq R$ and that $\forall x \exists y(xIy \wedge \forall z(yIz \Rightarrow z \leq x))$, respectively.

| NEGATION | STAR | PERP |
|---|---|---|
| Pre-minimal | No | $(\star)$ |
| Quasi-minimal | No | $(\star)$ and $R$ is symmetric |
| Intuitionistic | No | $(\star)$, $R$ is symmetric and reflexive |
| De Morgan | $x = x^{**}$ | $(\star)$, $R$ is symmetric and $(*)$ |
| Orthonegation | $x = x^*$ | $(\star)$, $R$ is symmetric, reflexive and $(*)$ |

## 3 Perp and Star in Relevance Logic

In this section, we apply the above propositions about perp and star in $DLL$ to the basic positive relevance logic $B_+$ and get one way in which relevance logic and modal logic can have a "happy marriage".

### 3.1 $B_N$ and Compatibility Meyer-Routley Frames

As you can see from above section, $K_-$ and $K_s$ are the weakest $DLL's$ that can be associated with perp and star semantics, respectively. In Došen [1986], he obtained the logic $N$ by adding the impossibility modality to the negation free fragment of propositional intuitionistic logic and shown the completeness of $N$ with respect to the class of compatibility frames. What is more interesting is that the "marriage" between the relation for negation and the partial relation for implication in $N$ is a "chemical" and harmonious one. Let $H$ be Heyting's propositional intuitionisitc logic. Define $R_N :=\leq \circ \leq^{-1}$. It is easy to show that the so-defined relation is antitonic in both places, and hence is a compatibility relation.

**Theorem 3.1** $\vdash_H A$ *iff* $A$ *is valid on all the above-defined compatibility frames.*[17]

In this section, we will take a similar enterprize in relevance logic. The relevant language and conventions are the same as in Routley and Meyer [1972] except that we will not include $t$ in the language $L$. For transparency,[18] we will sometimes denote principal arrows as $\leq$. As can be seen easily, $\leq$ here can be also regarded as $\vdash$ in $DLL$. We just list all the axioms and rules of the basic positive relevance logic $B_+$:

- $A \leq A$
- $A \wedge B \leq A, A \wedge B \leq B$
- $(A \to B) \wedge (A \to C) \leq A \to B \wedge C)$
- $A \leq A \vee B, B \leq A \vee B$
- $(A \to C) \wedge (B \to C) \leq (A \vee B \to C)$
- $A \wedge (B \vee C) \leq (A \wedge B) \vee C$
- $A, A \leq B \Rightarrow B$
- $A, B \Rightarrow A \wedge B$
- $A \leq B, C \leq D \Rightarrow B \to C \leq A \to D$

**Definition 3.2** *A Meyer-Routley frame* is a 3-tuple $F := \langle W, 0, R\rangle$ where $W$ is a nonempty set, $R$ a ternary relation,0 is the base world, and, if we define $x \leq y \equiv_{def} R0xy$, $R$ should additionally satisfies the following monotonicity conditions:

- $x \leq x$
- $x' \leq x$ and $Rxyz$ implies $Rx'yz$;
- $y' \leq y$ and $Rxyz$ implies $Rxy'z$;
- $z' \geq z$ and $Rxyz$ implies $Rxyz'$.

[17] Dunn [1996].

[18] The reason is obvious from the semantics below: the semantic consequence is defined in terms of truth preservation at the base world 0 in frames. Besides, we want to follow the notations from type theory and lambda calculus, with which relevance logic has a strong connection.

*A Meyer-Routley model* $M$ is a 4-tuple $M := \langle W, R, 0, v\rangle$ where $\langle W, R, 0\rangle$ is a Meyer-Routley frame and the interpretation $v$ is a function: $P \to 2^W$ (where $P$ is the set of propositional letters) satisfies the hereditary condition:

$$x \in v(p) \text{ and } x \leq y \text{ implies } y \in v(p) \text{ for arbitrary propositional letter } p.$$

Then we can extend the interpretation $v$ to model arbitrary formulas. Here we only give the semantic clause for $\to$-formulas

$$x \models A \to B \text{ iff for all } y, z \in W, Rxyz, \ y \models A \text{ implies } z \models B.$$

$A$ is *verified* on $v$ if $0 \models A$. $A$ is *valid* if $A$ is verified on all interpretations.

$\lhd$

**Theorem 3.3** *(Routley-Meyer)* $\vdash_{B_+} A$ *iff* $A$ *is valid on all Meyer-Routley frames.*

**Definition 3.4** *A compatibility Meyer-Routley frame* [19] is a a 4-tuple $\langle W, R, C, 0\rangle$ where $\langle W, R, 0\rangle$ is a Meyer-Routley frame and

$$xCy \text{ iff } \exists z(Rxyz).$$

It is easy to check that, if $xCy, x' \leq x$ and $y' \leq y$, then $Cx'y'$.

$\lhd$

What is the weakest relevance logic system that has such a semantics? In order to answer the question, we first extend our language $L$ to $L^{\bot\top}$ by adding propositional constant $\bot$ and $\top$ and define $\neg A \equiv_{def} A \to \bot$.[20] Here we treat $\neg$ as primitive in the language $L^{\bot\top}$. Let $B_N$ denote the logic system including all the axiom schemes and rules in $B_+$ in the extended language $L^{\top\bot}$ plus the following new axiom schemes:[21]

- $\neg A \wedge \neg B \leq \neg(A \vee B)$;
- $A \leq \top$;$\bot \leq A$;
- $\top \leq \neg\bot$;

and one more rule involving negation:

$$\text{(contraposition) } A \to B \Rightarrow \neg B \to \neg A.$$

Note that the first axiom scheme and the contraposition rule are redundant. But,in order for some comparisons in the last section, we keep them in the axiom scheme system. In addition to all those semantic clauses in $B_+$, we need two extra ones:

- $x \not\models \bot$ for all $x$
- $x \models \neg A$ iff $\forall y(xCy \to y \not\models A)$.

The second one is dispensable here. We can easily see that $x \models \neg A$ iff $x \models A \to \bot$,which is the reason why we define $xCy$ as $\exists z(Rxyz)$.

[19]We could also define compatibility Meyer-Routley frames in such a way that $R$ and $C$ are totally independent of each other. But such a kind of "marriage" is just "getting together" by name without much significance although it is legal. What we are really interested is the interaction between $R$ and $C$ so that we can explore the meaning of star.

[20]Another type of definition of $\neg$ through constants has been pursued in Mares [1993] in $R$. There he used the constant $f$.

[21]Recall that we did the same to DLL.

**Theorem 3.5** *(Soundness) Let $F$ be a compatibility Meyer-Routley frame. If $\vdash_{B_N} A$, then $F \models A$.*

Since there is a kind of subtlety in Henkin-style proofs of completeness in relevance logic, we will repeat the fundamental notions that can be found in Dunn [1986].

**Definition 3.6** A $\Pi$-*theory* $\Lambda$ is a non-empty set of formulas satisfying the following two closure conditions:

- If $\vdash_\Pi A \to B$ (which means $\Pi \vdash A \to B$) and $A \in \Lambda$, then $B \in \Lambda$;
- If $A \in \Lambda$ and $B \in \Lambda$, then $A \wedge B \in \Lambda$.

It is easy to see that $\top$ belongs to every theory. A $\Pi$ theory is *consistent* if it does not include all formulas in the language. A $\Pi$ consistent theory $\Lambda$ is *prime* if, for any $A \vee B \in \Lambda$, $A \in \Lambda$ or $B \in \Lambda$. It is a *regular* $\Pi$-theory if $\Pi \subseteq \Lambda$. A prime regular $\Pi$-theory is called *normal.* In canonical models, normal theories will function as base worlds. A $\Pi$ theory $\Lambda$ is *contrapositive* iff whenever $A \to B \in \Lambda$, $\neg B \to \neg A \in \Lambda$. It is *detached* iff whenever $A \to B, A \in \Lambda, B \in \Lambda$. It is *affixed* iff whenever $A \to B \in \Lambda$, $(B \to C) \to (A \to C) \in \Lambda$ and $(C \to A) \to (C \to B) \in \Lambda$. A $\Pi$ theory $\Lambda$ is *stable* iff it is consistent, affixed, contrapositive and detached.

$\lhd$

Now we use the "way-down" method due to Meyer to find a normal $B_N$ theory from any regular stable $B_N$ theory. For a regular stable $B_N$ theory $\Delta$, we recursively define a set $MT(\Delta)$[22] of formulas:

- $p \in MT(\Delta)$ iff $p \in \Delta$;
- $A \wedge B \in MT(\Delta)$ iff $A \in MT(\Delta)$ and $B \in MT(\Delta)$;
- $A \vee B \in MT(\Delta)$ iff $A \in MT(\Delta)$ or $B \in MT(\Delta)$;
- $A \to B \in MT(\Delta)$ iff (i) $A \to B \in \Delta$ (ii) If $A \in MT(\Delta)$, then $B \in MT(\Delta)$;
- $\neg A \in MT(\Delta)$ iff $A \notin MT(\Delta)$ and $\neg A \in \Delta$;
- $\bot \notin MT(\Delta), \top \in MT(\Delta)$.

Note that the clause for $\neg$-formulas can be derived from that for $\to$-formulas.

**Lemma 3.7** *Let $\Delta$ be a regular stable $B_N$ theory and $MT(\Delta)$ defined as above. Then $MT(\Delta) \subseteq \Delta$ and $MT(\Delta)$ is a normal $B_N$ theory.*

**Proof.** It is easy to show by induction on the complexity of $A$ that $MT(\Delta) \subseteq \Delta$. By definition, $MT(\Delta)$ is consistent and prime. It remains to show that it is regular, i.e. it is closed under all the inference rules and includes all the instances of axioms schemes in $B_N$. Here we just show that it (1) is closed under suffixing rule and (2) includes all the instances of $\neg A \wedge \neg B \leq \neg(A \vee B)$[23]. For the other cases, either the proofs have been covered in Dunn [1986] or can be shown similarly.

1. Let $A \to B \in MT(\Delta)$. We need to show that $(B \to C) \to (A \to C) \in MT(\Delta)$. Since $\Delta$ is stable and $A \to B \in \Delta$, $(B \to C) \to (A \to C) \in \Delta$. Assume that $B \to C \in MT(\Delta)$. We need to show that $A \to C \in MT(\Delta)$. First note that $A \to C \in \Delta$. It remains to show that $C \in MT(\Delta)$ whenever $A \in MT(\Delta)$. If

[22] $MT$ means metatruth.

[23] In fact, this case has been covered in Dunn [1986] if we think it in terms of $\bot$: $(A \to \bot) \wedge (B \to \bot) \leq (A \wedge B) \to \bot$. Here we prove that it belongs to $MT(\Delta)$ both as an illustration and to save energy for the logic $BN$ in the below section 3.3.

$A \in MT(\Delta)$, then $B \in MT(\Delta)$ for $A \to B \in MT(\Delta)$. Therefore $C \in MT(\Delta)$ for $B \to C \in MT(\Delta)$ by assumption.

2. First note that $\neg A \wedge \neg B \to \neg(A \vee B) \in \Delta$. Assume that $\neg A \wedge \neg B \in MT(\Delta)$. It follows that (1) both $\neg A \in \Delta$ and $\neg B \in \Delta$; (2) $A \notin MT(\Delta)$ and $B \notin MT(\Delta)$. We need to show that $\neg(A \vee B) \in MT(\Delta)$,i.e. $\neg(A \vee B) \in \Delta$ and $A \vee B \notin MT(\Delta)$. Obviously, by (2), $A \vee B \notin MT(\Delta)$. Moreover, $\neg A \wedge \neg B \in \Delta$. Since $\Delta$ is regular, $\neg(A \vee B) \in \Delta$. So, $\neg(A \vee B) \in MT(\Delta)$.

QED

The following definition is from Slaney [1987].

**Definition 3.8** *An immediate consequence* is any of the following five cases:

1. $A \wedge B$ is an immediate consequence of $A$ and $B$;
2. $B$ is an immediate consequence of $A \to B$ and $A$;
3. $\neg B \to \neg A$ is an immediate consequence of $A \to B$;
4. $(C \to A) \to (C \to B)$ is an immediate consequence of $A \to B$;
5. $(B \to C) \to (A \to C)$ is an immediate consequence of $A \to B$;

A *derivation* of formula $A$ from set $X$ in logic $B_N$ is a finite sequence of formulas, the last of which is $A$ and each of which is either a member of $X$, a theorem of $B_N$, or an immediate consequence of earlier ones.

$\lhd$

**Lemma 3.9** *Let $\nvdash_{B_N} A$. There is a consistent regular stable $B_N$ theory $\Delta_A$ such that $A \notin \Delta_A$.*

**Proof.** Without loss of generality, we assume that the cardinality of the language is countable. First we enumerate all the formulas $B_i$'s in the language $L^{\perp\top}$ such that each formula appears infinitely many times in the enumeration. Now we define a sequence of set of formulas recursively:

1. $\Delta_0 := \{\top\}$;
2. $\Delta_{n+1} := \Delta_n$ iff there is a derivation of $A$ from $\Delta_n \cup \{B_{n+1}\}$; $\Delta_{n+1} = \Delta_n \cup \{B_{n+1}\}$ otherwise.

Define $\Delta := \bigcup_n \Delta_n$. Note that $A \notin \Delta$ for otherwise there is a derivation of $A$ from some $\Delta_n$, which is impossible according to the recursive definition.

QED

**Corollary 3.10** *If $\nvdash_{B_N} A$, then there is a normal theory $\Pi_A$ such that $A \notin \Pi_A$.*

It is time to show the completeness. First we define the canonical model $M_\Pi$, which is determined by the normal $B_N$ theory $\Pi$:

- $W_\Pi$ consists of all prime $\Pi$-theories;
- $0_\Pi$ is the normal theory $\Pi$;
- $R_\Pi\Gamma\Delta\Theta$ iff, for any formulas $A$ and $B$, if $A \to B \in \Gamma$ and $A \in \Delta$, then $B \in \Theta$. It is easy to see that we can derive from this clause that $C_\Pi\Gamma\Delta$ iff $\neg^{-1}\Gamma \cap \Delta = \emptyset$.

Note that the canonical frame $F_\Pi := \langle W_\Pi, \subseteq, R_\Pi, C_\Pi \rangle$ is a compatibility Meyer-Routley frame. The canonical valuation is: $\Gamma \models_\Pi p$ iff $p \in \Gamma$. The proof of the Truth Lemma follows directly from the following well-known propositions in relevance logic:

**Lemma 3.11** *If $\Pi$ is a normal $B_N$ theory and $A \to B \notin \Pi$, then there is a prime $\Pi$ theory $\Gamma$ such that $A \in \Gamma$ but $B \notin \Gamma$.*

**Lemma 3.12** *If $\Sigma, \Gamma, \Delta$ are $\Pi$ theories, $R_\Pi \Sigma\Gamma\Delta$ and $A \notin \Delta$, then there are prime $\Pi$ theories $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$ such that $A \notin \Delta'$ and $R_\Pi \Sigma\Gamma'\Delta'$.*

**Lemma 3.13** *(Existence Lemma) Let $\Sigma$ be a prime $\Pi$ theory and $A \to B \notin \Sigma$. Then there are prime $\Pi$ theories, $\Gamma'$ and $\Delta'$ such that $R_\Pi \Sigma\Gamma'\Delta', A \in \Gamma'$ but $B \notin \Delta'$.*

**Lemma 3.14** *(Truth Lemma) For any formula $A$ and prime $\Pi$ theory $\Gamma$, $\Gamma \models_\Pi A$ iff $A \in \Gamma$.*

**Theorem 3.15** *(Completeness) If $A$ is valid on all compatibility Meyer-Routley frames, then $\vdash_{B_N} A$.*

**Proof.** Suppose that $\nvdash_{B_N} A$. By Corollary 3.10, there is a normal $B_N$ theory $\Pi$ such that $A \notin \Pi$. By appealing to the Truth Lemma, we have that there is a compatibility Meyer-Routley frame $F_\Pi$ such that $F \not\models_\Pi A$.

QED

## 3.2 $B_S$ and Being Star-Crossed

The treatment of star is also similar to that in *DLL*.

**Definition 3.16** A compatibility Meyer-Routley frame $F := \langle W, R, C, \leq \rangle$ is *star crossed* if $F$ additionally satisfies the following condition:

$$(*) : \forall x \exists y \forall z (xCz \leftrightarrow z \leq y).$$

$\lhd$

Now what principles can be added to $B_N$ to make the frames appropriate to this logic be star crossed? The answer is the same as that to *DDL*: $\neg(A \wedge B) \to \neg A \vee \neg B$ and $\neg\top \to \bot$. Let $B_S$ be the logic $B_N$ plus these two axiom schemes. Since $B_S$ has the full power of $B_+$, the following variant of Pair Extension Lemma is just immediate.

**Lemma 3.17** *(Pair Extension Lemma 2) Let $\Sigma$ be a $\Pi$ theory, $\Delta$ is closed under disjunction and $\Sigma \cap \Delta = \emptyset$. Then there is a $\Sigma' \supseteq \Sigma$ such that $\Sigma' \cap \Delta = \emptyset$ and $\Sigma'$ is a prime $\Pi$ theory.*

**Theorem 3.18** $\vdash_{B_S} A$ *iff $A$ is valid on all star-crossed compatibility Meyer-Routley frames.*

**Proof.** In order to show the left-right direction, we only need to check that all instances of the above two new axioms are valid on any star crossed compatibility frames, which is easy.

Now for the more difficulty direction. It is easy to see that $\neg(A \wedge B) \to \neg A \vee \neg B \in MT(\Delta)$ and $\neg\top \to \bot \in MT(\Delta)$ for any regular stable $B_S$ theory $\Delta$. Therefore, if $\nvdash_{B_S} A$, then there is a normal $B_S$ theory $\Pi$ such that $A \notin \Pi$.

By appealing to the above lemma, we have that for any $\Pi$ prime theory $\Sigma$, there is a $\Pi$ prime theory $\Theta$ such that $\Sigma C_\Pi \Theta$ and, if we define $\Sigma^* := \{A : \neg A \notin \Sigma\}$, then $\Sigma^*$ is also a prime $\Pi$ theory and $\Sigma C_\Pi \Lambda$ iff $\Lambda \subseteq \Sigma^*$. Here we only show the last part $\Sigma C_\Pi \Lambda$ iff $\Lambda \subseteq \Sigma^*$.

Assume that $\Sigma C_\Pi \Lambda$. Let $A \in \Lambda$. By the definition of $C_\Pi$, $\neg A \notin \Sigma$. That is to say, $A \in \Sigma^*$. Conversely, assume that $\Lambda \subseteq \Sigma^*$. Let $\neg A \in \Sigma$. It follows that $A \notin \Sigma^*$ and hence $A \notin \Lambda$. So $\Sigma C_\Pi \Lambda$.

QED

## 3.3 Conservation

In the above we work with the language $L^{\top\bot}$ which includes two constants $\top$ and $\bot$. In the following we will work with the language $L$ without these constants. It is natural to ask what are the $L$ fragments of $B_N$ and $B_S$, respectively? Let $BN$ be the logic $B_+$ in the language $L$ plus the axiom scheme $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$, the more important axiom scheme:

$$(\delta)\text{: } (A \rightarrow C) \wedge \neg B \rightarrow (A \vee B) \rightarrow C$$

and one extra rule:

$$A \rightarrow B \Rightarrow \neg B \rightarrow \neg A.$$

The definition of compatibility Meyer-Routley frames is the same as that in the previous sections. But the definition of compatibility Meyer-Routley *models* is different from that before. First we have to delete all the semantic clauses for formulas involving $\bot$. Secondly, the modelling condition for $\neg$ formulas can not derived from that for $\rightarrow$ formulas. So we have to add a new one for $\neg$-formulas:

- $x \models \neg A$ iff $\forall y(xCy \rightarrow y \not\models A)$.

**Theorem 3.19** *Let $F$ be a compatibility Meyer-Routley frame. If $\vdash_{BN} A$, then $A$ is valid on all compatibility Meyer-Routley frames.*

**Proof.** The only necessary thing is to show that, for any compatibility Meyer -Routley frame $F$: $(\delta)$ is valid on $F$. Suppose that $x \models (A \rightarrow C) \wedge \neg B$. We need to show that $x \models A \vee B \rightarrow C$. Let $Rxyz$ and $y \models A \vee B$. Since $x \models \neg B$, $y \not\models B$. Therefore $y \models A$. This implies that $z \models C$ for $x \models A \rightarrow C$. So $x \models A \vee B \rightarrow C$.

QED

**Theorem 3.20** *$\nvdash_{BN} A$ iff $A$ is not valid on all compatibility Meyer-Routley frames.*

Before we show the main theorem, we will prove several lemmas and then the main theorem will follow directly from these lemmas. We will use a similar method to that in Section 3.1. Assume that $\nvdash_{BN} A$. Then there is a regular stable $BN$ theory $\Delta$ such that $A \notin \Delta$ (note that $\Delta$ is consistent of course). Define $MT(\Delta)$ in the same way except that we don't need the two clauses about $\top$ and $\bot$.

**Lemma 3.21** *$MT(\Delta) \subseteq \Delta$ and $MT(\Delta)$ is a normal $BN$ theory.*

**Proof.** Here we only check that $(A \rightarrow C) \wedge \neg B \rightarrow (A \vee B) \rightarrow C \in MT(\Delta)$. First note that it is indeed in $\Delta$. Assume that $(A \rightarrow C) \wedge \neg B \in MT(\Delta)$. We need to show that $A \vee B \rightarrow C \in MT(\Delta)$. Similarly, it is obvious that $A \vee B \rightarrow C \in \Delta$. Let $A \vee B \in MT(\Delta)$. Since $\neg B \in MT(\Delta)$, $B \notin MT(\Delta)$. Therefore $A \in MT(\Delta)$. And hence $C \in MT(\Delta)$. So $A \vee B \rightarrow C \in MT(\Delta)$. What this claim says is that if $\nvdash_{BN} A$, there is a normal $BN$ theory $\Pi$ such that $A \notin \Pi$.

QED

We can define the canonical model determined by $\Pi$ as above except that we need an independent clause for $C_\Pi$: $\Gamma C_\Pi \Delta$ iff $\neg^{-1}\Gamma \cap \Delta = \emptyset$, which can not be derived from the definition of $R_\Pi$. Define: $R^C_\Pi \Gamma\Delta\Theta \equiv_{def} R_\Pi \Gamma\Delta\Theta$ and $C_\Pi\Gamma\Delta$. Obviously, the so-defined canonical frame $F_\Pi := \langle W_\Pi, R^C_\Pi, C_\Pi, \subseteq\rangle$ is a compatibility Meyer-Routley frame. The modelling conditions are a little different that in Section 3.1:

- $\Gamma \models_\Pi A \to B$ iff $\forall \Delta, \Theta$, if $R^C_\Pi \Gamma\Delta\Theta$ and $\Delta \models_\Pi A$, then $\Theta \models_\Pi B$ (note that here we have strenghened the accessibility relation from $R_\Pi$ to $R^C_\Pi$);
- $\Gamma \models_\Pi \neg A$ iff $\forall \Delta (C_\Pi \Gamma\Delta \to \Delta \not\models_\Pi A)$.

**Lemma 3.22** *Let $\Pi$ be the normal BN theory. If $A \to B \notin \Pi$, then there is a $\Pi$ prime theory $\Gamma$ such that $A \in \Gamma$ but $B \notin \Gamma$.*

**Lemma 3.23** *Let $\Sigma$ be a prime $\Pi$ theory and $A \to B \notin \Sigma$. Then there are two $\Pi$ theories $\Gamma$ and $\Delta$ such that $R_\Pi \Sigma\Gamma\Delta$, $A \in \Gamma$ and $B \notin \Delta$.*

**Proof.** The proof here is similar to that of lemma 4 in Priest and Sylvan [1992].

QED

**Lemma 3.24** *(Squeezing Lemma) If $\Sigma$ is a prime $\Pi$ theory, and $\Gamma, \Delta$ are $\Pi$ theories, $R_\Pi \Sigma\Gamma\Delta$ and $D \notin \Delta$, then there are prime $\Pi$ theories $\Gamma', \Delta'$ such that $\Gamma \subseteq \Gamma', \Delta \subseteq \Delta', D \notin \Delta'$ and $R^C_\Pi \Sigma\Gamma'\Delta'$.*

**Proof.** First note that by appealing to Pair Extension Lemma we can get a prime $\Pi$ theory $\Delta'$ such that $D \notin \Delta', R_\Pi\Sigma\Gamma\Delta'$ and $\Delta \subseteq \Delta'$. The next step is the most crucial in the whole proof. Let

- $\Theta_1 := \{A : \exists B (B \notin \Delta' \wedge A \to B \in \Sigma)\}$;
- $\Theta_2 := \{A : \neg A \in \Sigma\}$.
- $\Theta := \Theta_1 \cup \Theta_2$.

Now we show that $\Theta$ is closed under disjunction. First it is easy to see that both $\Theta_1$ and $\Theta_2$ are closed under disjunction. In order to show the closure of $\Theta$, it suffices to show that, for any $A_1 \in \Theta_1$ and any $A_2 \in \Theta_2$, $A_1 \vee A_2 \in \Theta$. Since $A_1 \in \Theta_1$, there is a $B_1$ such that $B_1 \notin \Delta'$ and $A_1 \to B_1 \in \Sigma$. Moreover, $\neg A_2 \in \Sigma$. It follows that $\neg A_2 \wedge (A_1 \to B_1) \in \Sigma$. Since $(A_1 \to B_1) \wedge \neg A_2 \to ((A_1 \vee A_2) \to B_1) \in BN \subseteq \Pi$, $(A_1 \vee A_2) \to B_1 \in \Sigma$. By the definition of $\Theta_1$, $A_1 \vee A_2 \in \Theta_1 \subseteq \Theta$.

Next we show that $\Gamma \cap \Theta = \emptyset$. Suppose that there is a $A \in \Gamma \cap \Theta$. Then there is a $B$ such that $A \to B \in \Sigma, B \notin \Delta'$ and $A \in \Gamma$. Since $R_\Pi\Sigma\Gamma\Delta'$. Then we get that $B \in \Delta'$. This is a contradiction.

By Pair Extension Lemma, we obtain a prime $\Pi$ theory $\Gamma'$ such that $\Gamma \subseteq \Gamma', \Gamma' \cap \Theta = \emptyset$. It is easy to check that $R^C_\Pi \Sigma\Gamma'\Delta'$.

QED

Combining above three, we can get the following Existence Lemma:

**Lemma 3.25** *(Existence Lemma for $\to$-formulas) Let $\Sigma$ be a prime $\Pi$ theory and $A \to B \notin \Sigma$. Then there are two prime $\Pi$ theories $\Gamma$ and $\Delta$ such that $R^C_\Pi \Sigma\Gamma\Delta$, $A \in \Gamma$ and $B \notin \Delta$.*

**Lemma 3.26** *(Existence Lemma for $\neg$ formulas) If $\Sigma$ is a prime $\Pi$ theory and $\neg A \notin \Sigma$, then there is a prime $\Pi$ theory $\Gamma$ such that $A \in \Gamma$ and $\Sigma C_\Pi \Gamma$.*

**Proof.** First note that $\neg^{-1}\Sigma$ is closed under disjunction. Let $[A)$ denote the $\Pi$ theory generated by $A$. In order to apply Pair Extension Lemma, we only need to show that $\neg^{-1}\Sigma \cap [A) = \emptyset$. Suppose not. Then there is a formula $B$ such that $\vdash_\Pi A \to B$ and $\neg B \in \Sigma$. It follows that $\neg A \in \Sigma$, which contradicts our assumption that $\neg A \notin \Sigma$.

QED

**Lemma 3.27** *(Truth Lemma)* $\Gamma \models_\Pi A$ *iff* $A \in \Gamma$.

Now we return to the proof of the main theorem. Since $A \notin \Pi$, we have that $F_\Pi, 0_\Pi \not\models_\Pi A$. So $A$ is not valid on all compatibility Meyer-Routley frames.

**Corollary 3.28** *(Conservation 1) Let A be a formula in the language L. Then* $\vdash_{BN} A$ *iff* $\vdash_{B_N} A$. *That is to say,* $B_N$ *is conservative over BN with respect to the language L.*

Since the binary compatibility relation is defined from the ternary relation $R$, the following conservation is immediate:

**Corollary 3.29** *(Conservation 2)Let* $L_+$ *be the labguage L without the connective* $\neg$ *and A be a formula in* $L_+$. *Then* $\vdash_{B_+} A$ *iff* $\vdash_{BN} A$. *In other words, BN is a negative conservative extension of* $B_+$.[24]

Next we will show another expected conservation result after the above one. Let $BS$ denote the logic $BN$ plus the axiom scheme:

$$\neg(A \wedge B) \to \neg A \vee \neg B.$$

The proof of the conservation of $B_S$ over $BS$ is similar to the above one.

**Corollary 3.30** *(Conservation 3) Let A be a formula in the language L. Then* $\vdash_{BS} A$ *iff* $\vdash_{B_S} A$. *That is to say,* $B_S$ *is conservative over BS with respect to the language L.*

If we take $BM$ and its frames (called $BM$ frames where $R$ and $*$ are independent of each other) as primitive[25], $BS$ can be taken as an extension of $BM$. Similarly, we can show that

**Theorem 3.31** $\vdash_{BS} A$ *iff A is valid on all the BM frames satisfying:*

$$(\star) : \forall xy(y \leq x^* \leftrightarrow \exists z(Rxyz))$$

After applying Dunn's translation between perp and star, we get exactly the condition: $\forall xy(xCy \leftrightarrow \exists z(Rxyz))$ that we have imposed in the definition of compatibility Meyer-Routley frames.

Now it remains to explain the meaning of the Routley star in our context. In fact, according to the above analysis, for any point $x$ in the compatibility Meyer-Routley frame, $x^*$ is defined by the following first order formula $S_x(y)$ which involves only $R$ and $\leq$:

$$\forall z(\exists w(Rxzw) \leftrightarrow z \leq y).$$

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[24] In literature, Meyer's favorite $CB$ and Routley' $BM$ are also negative conservative over $B_+$.

[25] Priest et al. [1992].

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