Time in the *Critique of Pure Reason*

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This note is dedicated to Frank, Jeroen and Martin, whose intense engagement with semantics first opened my eyes to the beauty of a topic that combines formal precision with philosophical reflection and cognitive experimentation. In particular the problem of the cognitive import of formal semantic representations has been the subject of countless discussions, especially with Martin. Tense and aspect seemed a promising area in which to look for semantic representations with cognitive import. The present essay investigates how to reconcile a cognitive desideratum, finiteness, with the expressive richness required by tense and aspect: completeness and incompleteness of events, granularity, continuous vs. discrete change etc. Because Kant had considered several of these issues in the *Critique of Pure Reason*, this essay approaches the issues just mentioned bt describing a model that satisfies Kant's synthetic a priori principles for time. For those unmoved by Kantian considerations, it suffices to take a brief look at the principles listed below, after which the technical development should make sense.

1 Principles for time

The *Critique of Pure Reason* [2] is interesting for those studying the cognitive development of fundamental concepts and capacities, in this case time, because Kant dissects a capacity into components, which work together under the guidance of a consistency monitor, what Kant calls the 'transcendental unity of apperception'. In young children one often finds that what Kant analyses conceptually, is in fact still dissociated in the child's brain. With regard to time, the following has been observed.
(i) There is no reliable correlation between causal and temporal order (i.e. children do not object to backwards causation) and children at chance at inferring temporal order of hidden events from causal premises; in Kant's terminology one would say that these children still lack the *category of causality*.
(ii) The order of events is sometimes encoded, but generally not accessible to reasoning (e.g. children find it difficult to recite an event sequence in reverse order); in Kant's

terminology this would be glossed by saying that children have the *form of intuition* of time, but not yet the *capacity to judge* in this particular domain.
Examples could be multiplied, but here we focus on how Kant dissects time, and puts it back together again.

In the *Critique of Pure Reason* Kant offers two extended discussions of the nature of time as a form of intuition: in the *Transcendental Aesthetic*, and in the *Analogies of Experience*, which explain the role the three relational Categories – substance, causality and community – play in time-determination, that is, the assignment of definite positions in time to events. The *Transcendental Aesthetic* lists a number a synthetic a priori principles for time, such as

1. (A31/B47) Time has only one dimension; different times are not simultaneous, but successive.

2. 'The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation time must there be given as unlimited.' (A32/B47-8)

3. (A32/B47) Different times are only part of one and the same time.

4. (ibidem) Instants arise only as boundary points

and in the *Analogies*, whose purpose it is to show

> As regards their existence, appearances stand a priori under rules of the determination of their relation to each other in *one* time.

we find in addition

1. (A177/B219): 'The three modi of time are *persistence*, *succession* and *simultaneity*' where persistence is explained as

    > Only through that which persists does existence in different parts of the temporal series acquire a magnitude, which one calls duration. For in mere sequence alone existence is always disappearing and beginning, and never has the least magnitude. Without that which persists there is therefore no temporal relation.

2. (A209/B254) Time does not consist of smallest parts [i.e. is infinitely divisible but is not composed of points]

3. It follows that change is always continuous, and that the 'boundary points' of A32/B47 must have non-empty parts

As to the source of our knowledge of these synthetic a priori principles, Kant writes

> This a priori necessity [of time] also grounds the possibility of apodeictic principles of relations of time, or axioms of time in general. (A31/B47)

The *a priori* character of time stems from the fact that time cannot be perceived, hence cannot have been acquired through experience. We do of course make temporal judgements of precedence and simultaneity, but these already presuppose such judgements.

One would like to know in greater detail how the 'necessity of time' leads to axioms of time like those listed above, but one can see that a formal axiomatic approach might be able to describe how temporal judgements of precedence and simultaneity must depend boundaries must upon other such judgements, leading to axioms of the form $\forall \bar{x}(\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$, where φ, ψ are Boolean combinations of temporal relations. When we begin reflecting on what the range of the bound variables could be, questions arise. The domain of quantification cannot be a set of temporal instants, because time is not composed of instants; in fact there will be few such instants, since these can only be given as boundaries separating two parts of time, and hence must be extended in time, since time has no smallest parts. On the other hand, one may add new boundaries indefinitely, since time is infinitely divisible. Furthermore, Kant repeatedly emphasis topological properties of time: persistence as opposed to discreteness), continuity of change etc., all of which must be explained in a setting where few points are available. It is the interplay between order-theoretic and topological notions in Kant's theory of time that makes the task of formalising it daunting, but also interesting.

Our strategy will be as follows. We first present a longish list of order-theoretic axioms, and show that first order models can be expanded to second order models in which there exists a one dimensional time with instants defined as boundaries. We then consider persistence and infinite divisibility. In the end we arrive at a model for all of Kant's synthetic a priori principles for time.

2 Order-theoretic axioms

For Kant, the main question concerning time is one whose answer is announced in (A177/B219):

> As regards their existence, appearances stand a priori under rules of the determination of their relation to each other in *one* time.

Formally, this means that there is a function that maps appearances to their position in time; we shall call this mapping the *tenure* function, and if a is an appearance,[1]

[1] A combination of perceptual features, which as yet has not achieved object-status.

$e = ten(a)$ will be called an *event*. The second and third Analogies argue that precedence and simultaneity are the relevant relations between events. This suggests to take as primitive a relation P between events, representing 'total precedence', satisfying axioms such as irreflexivity and transitivity, as suggested by Russell [3], Walker [5], Kamp [1] and Thomason [4]. However, a moment's reflection will show that this concept of precedence fits Hume, who conceived of causal chains as discrete, rather than Kant, who did not. Kant viewed effects in a causal chain as *alterations* – changes of state – and argued that such effects will generally be simultaneous with their causes, and moreover that *natura non facit saltus*: changes of state are continuous and themselves take time. This means that the left and right boundaries of an event (say the result of a state change), even though somewhat indefinite, will overlap with another event (the cause of that change). The appropriate representation of precedence is therefore the pair of predicates $B(e, d)$ for 'e begins after d (begins)' and $E(e, d)$ for 'e ends before d (ends)'. In addition we have a reflexive and symmetric predicate O for 'overlap', which in certain circumstances one can take to be transitive as well. It will be useful to have the axioms for O, B, D in geometric form, which means that we must replace negations $\neg O$ etc. by positive predicates $\breve{O}$ etc. A structure $\mathcal{W} := (W; O, \breve{O}, E, \breve{E}, B, \breve{B})$, where W is a set of events, will be called an *event structure*. For ease of exposition we introduce a notational convention

Definition 1. *Define the relation $\preceq$ by*

$$b \preceq a \Leftrightarrow \breve{B}(a, b) \wedge \breve{E}(a, b) \wedge O(a, b).$$

If $b \preceq a$ holds, we say that a covers *b.*

Definition 2. *For event structures $\mathcal{W} := (W; O, \breve{O}, E, \breve{E}, B, \breve{B})$ we adopt the following axioms AX_0*

1. $E(a, b) \wedge \breve{E}(a, b) \rightarrow \bot$
2. $E(a, b) \vee \breve{E}(a, b)$ [excluded middle for $\breve{E}$]
3. $B(a, b) \wedge \breve{B}(a, b) \rightarrow \bot$
4. $B(a, b) \vee \breve{B}(a, b)$ [excluded middle for $\breve{B}$]
5. $\breve{E}(a, b) \vee \breve{E}(b, a)$ [implies $\breve{E}$ is reflexive]
6. $\breve{B}(a, b) \vee \breve{B}(b, a)$ [implies $\breve{B}$ is reflexive]
7. $\breve{E}(a, b) \wedge \breve{E}(b, c) \rightarrow \breve{E}(a, c)$ [$\breve{E}$ is transitive]

8. $\breve{B}(b,a) \wedge \breve{B}(c,b) \rightarrow \breve{B}(c,a)$ [$\breve{B}$ is transitive]
9. $O(a,b) \wedge \breve{O}(a,b) \rightarrow \bot$
10. $O(a,b) \vee \breve{O}(a,b)$
11. $O(a,a)$
12. $O(a,b) \rightarrow O(b,a)$
13. $O(a,b) \rightarrow \exists c(a \preceq c \wedge b \preceq c)$
14. $O(c,a) \wedge O(c,b) \wedge \breve{B}(a,b) \wedge \breve{E}(a,b) \rightarrow O(a,b)$
15. $O(c,a) \wedge O(c,b) \wedge \breve{B}(a,b) \wedge \breve{B}(b,a) \rightarrow O(a,b)$ [similarly with $\breve{E}$ replacing $\breve{B}$]

The system AX_l ('l' for 'linearity') comprises AX_0 plus

16 $\breve{E}(a,b) \wedge \breve{B}(a,b) \rightarrow O(a,b)$ [implies $\breve{O}(a,b) \rightarrow \mathcal{B}(a,b) \wedge E(a,b)$]

Axiom 13 expresses that parts of time come from the whole of time. The condition '$O(a,b)$' has been added because of Kant's subtle notion of point: points are not 'smallest parts' but divisible, and this makes it doubtful whether bisection into disjoint parts exist.

The last three axioms embody various approximations to linearity, The difference between axiom 2 and the other two is that the latter require that the events involved are comparable in time (this is what the antecedent $O(c,a) \wedge O(c,b)$ expresses, whereas 2 has no such requirement. Although the axiom system AX_l allows more straightforward proofs of the results presented here, we will work in the much more parsimonious system AX_0, because linearity has to be justified, at least in the Kantian context.

The transitivity axioms, for instance 8, can be motivated as follows. We do not assume linearity of time. If each of a,b,c lies in a different time line, then axiom 8 is trivially true. Assume a,b are on the same timeline, and c is not; then $\breve{B}(c,a)$ is trivially true. The remaining case is the one where a,b,c lie on the same timeline, and here the axiom clearly holds.[2]

In keeping with Kant's constructivism, the system is formulated in a restricted language (geometric logic) where classical and intuitionistic entailment coincide. This implies that axioms 2, 4 are harmless additions; what can be proven using these axioms can also be proven without them.[3]

[2]One can also make a case for the transitivity of B, E, but these relations are irreflexive, which will turn out to be an important drawback, because they make the topological methods used here inapplicable.

[3]The technically correct formulation is: every geometric formula provable in AX_0 is provable with using 2, 4 .

Definition 3. *AX_0^i ('i' for 'intuitionistic') consists of AX_0 minus 2, 4.*

Lemma 1. $E(a,b) \to \breve{E}(b,a)$.

PROOF. Assume $E(a,b)$, then by axiom 1 $\neg\breve{E}(a,b)$, whence by axiom 5, $\breve{E}(b,a)$. □

Lemma 2. *The relation $\preceq$ as defined in 1 is transitive and reflexive.*

PROOF. From axioms 14, 11, 6, 5, 8, 7. □

3 Topological properties

Several properties of time listed by Kant concern its topology: time is infinitely divisible, time does not consist of instants, time is not 'mere sequence' but 'persistent', alterations are always gradual and continuous, and instants arise only as boundary points (which may themselves be extended).
This list makes clear that Kantian time cannot be represented by the real number line, with the topology generated by the open intervals. We will derive first more useful topologies from the predicates introduced in the axiomatisation.

The relations $\breve{B}(a,b), \breve{E}(a,b)$ are reflexive and transitive, and thus lend themselves to the following construction

Definition 4. *Let R be a reflexive and transitive relation on a set X. $G \subseteq X$ is R-*upwards closed *if $a \in G, R(a,b) \Rightarrow b \in G$. We omit reference to R when it is clear from the context. Arbitrary unions and intersections of upwards closed sets are again upwards closed. The upwards closed sets will be called* open*, and their complements, the downwards closed sets, will be called* closed*. The collection of open sets is called the* Alexandroff topology.

The relations $\breve{B}(a,b), \breve{E}(a,b)$ define Alexandroff topologies, as do the upwards closed sets of the relation $\preceq$ defined in 1. However, for our purposes the downwards closed subsets of $\preceq$ are more important.

As we shall see, all three topologies have a temporal meaning: $\breve{E}$ represents past, (the closed sets of) $\preceq$ present and $\breve{B}$ future. Our ultimate aim is to show that the set of events can be given the structure of a one dimensional continuum, which may have some instants, all of which arise as boundaries; but there are some surprises along the way.

We argue as follows. A temporal boundary in an event structure $\mathcal{W}$ determines a set of events *Past* in the past of that boundary, and likewise a set of events *Fut* which all lie in the future of the boundary. We have that $a \in$ *Past*$, \breve{E}(a,b) \Rightarrow b \in$ *Past* and

$a \in$ ***Fut***, $\breve{B}(a, b) \Rightarrow b \in$ ***Fut***. Using the terminology introduced above, we say that *Past* is $\breve{E}$ open, while *Fut* is $\breve{B}$ open. Furthermore, under the given interpretation $a \in$ *Past* and $b \in$ *Fut* implies $\breve{O}(a, b)$, which in turn implies that *Past* and *Fut* are (set theoretically) disjoint. The complement of *Past* $\cup$ *Fut* can be viewed as a representation of the temporal boundary between *Past* and *Fut*, which we might as well call the present (*Pres*). These considerations are summarised in the following definition.

Definition 5. *Given an event structure* $\mathcal{W} := (W; O, \breve{O}, E, \breve{E}, B, \breve{B})$*, an* instant *in* $\mathcal{W}$ *is a triple* $(\textit{Past}, \textit{Pres}, \textit{Fut})$.

Lemma 3. *Pres, the complement of Past* $\cup$ *Fut, is* $\preceq$*-closed.*

We now have to investigate whether the axioms AX_0 imply that the collection of triples $(\textit{Past}, \textit{Pres}, \textit{Fut})$ can be linearly ordered. A reasonable guess is that the linear order $<$ must be defined by

Definition 6. $(\textit{Past}, \textit{Pres}, \textit{Fut}) < (\textit{Past}', \textit{Pres}', \textit{Fut}')$ *if Past* $\subseteq$ *Past*$'$.

This suggestion doesn't work for all instants. In fact, given *Past* only, *Pres* can be chosen independently and we have a structure that is at least two-dimensional. A linear order can be obtained from a special kind of instants, namely those instants $(\textit{Past}, \textit{Pres}, \textit{Fut})$ such that $\{< a, b >| \ a \in \textit{Past}, b \in \textit{Fut}\}$ is a maximal open[4] subset of $\{< a, b >| \ \breve{O}(a, b)\}$. This suggestion can be made fully precise, although we shall refrain from doing so. We note that the maximality of $\{< a, b >| \ a \in \textit{Past}, b \in \textit{Fut}\}$ entails the minimality of *Pres*. We therefore define

Definition 7. *An instant* $(\textit{Past}, \textit{Pres}, \textit{Fut})$ *is a* thin boundary *if* $\{< a, b >| \ a \in \textit{Past}, b \in \textit{Fut}\}$ *is a maximal open subset of* $\{< a, b >| \ \breve{O}(a, b)\}$.

A thin boundary may have an empty present; we will later formulate a topological condition equivalent to non-emptyness of the present. At this point the reader may wonder: why 'thin' boundary, and not 'thinnest'? This has to do with the Kantian dictum that time has no smallest parts, hence boundaries – the only kind of instants for Kant – have no smallest parts either. Boundaries are analogous to points as considered in set theoretic topology, in that they are closed sets, but they must be thought of as extended.

Theorem 1. *The set of thin boundaries in an event structure has a linear order as defined in definition 6.*

[4]Technically: open in the product topology generated by $\breve{E} \wedge \breve{B}$.

Lemma 4. *If $(Past, Pres, Fut)$ is a thin boundary, then for all $c, d \in Pres : O(c, d)$, hence on Pres O is an equivalence relation, which we may call* simultaneity.

Russell, using a simpler set of axioms involving only 'total precedence' and 'overlap', defined instants as maximal sets of pairwise overlapping events, and showed that these can be linearly ordered. Since for Kant time is not made up of smallest parts, we should not think of time as the set of thin boundaries. Rather boundaries are limitations of parts of time, and these parts may well be empty in the sense of not containing any boundaries. How to reconcile this with the 'persistence' of time that Kant discusses in the first Analogy will be considered below.

But what of the instants that are not thin boundaries, i.e. instants of which the *Pres* component is slightly overweight? These *Pres* will contain events that are not simultaneous (in the sense transitive O), hence instants $(Past, Pres, Fut)$ containing such *Pres* can be decomposed into several thin boundaries. The instant with the 'fat' present may then be viewed as an instant for which it is uncertain with which thin boundary it is simultaneous. Such 'fat' instants clearly belong to subjective, not to objective time.

Before moving on to a different topic, we note that we now have a representation of Kant's version of the infinity of time

> The infinitude of time signifies nothing more than that every determinate magnitude of time is only possible through limitations of a single time grounding it. The original representation time must there be given as unlimited. (A32/B47-8)

since the triples $(\emptyset, \emptyset, W)$ and $(W, \emptyset, \emptyset)$ are like $-\infty, +\infty$ on the real number line: every ordinary thin boundary lies strictly between these two extremes.

3.1 Infinite divisibility

At any given stage we can have constructed only finitely many boundary points, but there is no fixed bound on how many boundary points can be constructed. By splitting events we can introduce new boundaries and make existing ones thinner:

Definition 8. *Let $c \in W$ be given. A pair $< a, b >$ splits c if $\breve{O}(a, b)$, $O(c, a)$, $O(c, b)$, $\breve{B}(a, b)$,*[5] *$\breve{B}(c, a)$, $\breve{E}(c, b)$.*

Now suppose we have finite event structures $\mathcal{W}_1, \mathcal{W}_2$ such that the second contains splittings of some events in the first, and perhaps also events d such that for all $e \in W_1$: $\breve{E}(d, e)$, which represent the flow of time toward the future. We can represent the splitting and temporal flow toward the future as follows.

[5]This is a convention which says that the first coordinate is in the past of the second.

Definition 9. *A function $f : \mathcal{W}_2 \longrightarrow \mathcal{W}_1$ is* continuous *if it preserves $O, \breve{B}, \breve{E}$.*

If $< a, b >$ splits c, we may define f on these events by $f(a) = c = f(b)$. Infinite divisibility may now be represented by an 'inverse sequence'

$$\ldots\ldots \mathcal{W}_3 \longrightarrow_{f_3} \mathcal{W}_2 \longrightarrow_{f_2} \mathcal{W}_1 \longrightarrow_{f_1} \mathcal{W}_0.$$

Here we see how thin boundaries can be split: on the one hand, a thin boundary consists of simultaneous events and behaves like a point; on the other hand, a thin boundary has extension and can be subdivided.

3.2 Persistence

Kant writes in the first Analogy that a 'mere sequence' of events, i.e. a sequence in which no two events overlap, does not support duration. His way to rule out such sequences is to posit that the category of substance allows one to assume time is persistent, and does not come in fits and starts. This idea can be represented in the model, but a brief indication must suffice here.

Definition 10. *A topological space is* connected *if it cannot be written as the union of two non-empty disjoint open sets.*

Lemma 5. *An event structure is connected in the topology generated by $\breve{B} \vee \breve{E}$ if and only if all boundaries have non-empty Pres.*

This lemma gives a condition under which all change is continuous. But combined with axiom 13 it says more

Lemma 6. *In finite connected event structures $\mathcal{W}$ there exists an event $w \in W$ such that for all Pres: $w \in$ Pres.*

Definition 11. *A topological space is* ultraconnected *if any two non-empty closed sets have non-empty intersection.*

On finite event structures ultraconnectedness is equivalent to the property enunciated in lemma 6. Observe that w cannot be an element of *Past* or *Fut*; in that sense it is a permanent substrate for all temporal change, as required by Kant in the first Analogy. We close by providing an equivalent formulation of this property, which shows it to be equivalent to one that expresses that time cannot be viewed as a sum of parts.

Definition 12. *Let $\mathcal{W}$ be an event structure, with the topology generated by $\breve{B} \vee \breve{E}$. A collection of open sets $\mathcal{C}$ is a* cover *of $\mathcal{W}$ if*

$$\forall a \in W \exists O \in \mathcal{C} \exists c \in O (a \preceq c).$$

Lemma 7. *In finite connected event structures the following holds (w.r.t. the topology generated by $\breve{B} \vee \breve{E}$): every cover must contain the whole space W as one of its open sets.*

PROOF. If $\mathcal{C}$ is a cover, there must be $O \in \mathcal{C}$ with $w \in O$; but this holds only if $O = W$. $\square$

These considerations are not confined to finite structures, they hold as well for infinite structures under mild saturation conditions, or for profinite structures (inverse limits of finite structures). But the purpose of this note was to show that the aspects of time that Kant carefully distinguished can both be separated axiomatically, and shown to be jointly consistent.

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