An appendix to DPL

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For Jeroen Groenendijk & Martin Stokhof

1 Introduction

I am afraid this paper will appeal to only a select group of readers. One has to be well acquainted with Fitch style natural deduction for static predicate logic, one has to be well acquainted with dynamic predicate logic, and then, most importantly, one should be curious to know if and how a Fitch style natural deduction system for static predicate logic can be made dynamic. [1]

Of course, the most important step in going from a static to a dynamic proof system is to replace the elimination rule for the static existential quantifier pictured below[2] by an elimination rule that fits the dynamic quantifier.

Static elimination rule for $\exists$

$$
\begin{array}{ccc}
\vdots & \vdots & \\
l & \exists x\varphi & \\
\vdots & \vdots & \\
m & [{}^{a}/x]\varphi \to \psi & \\
\vdots & \vdots & \\
n & \psi & E_{\exists}, l, m
\end{array}
$$

Dynamic elimination rule for $\exists$

$$
\begin{array}{ccc}
\vdots & \vdots & \\
m & \exists x\varphi & \\
\vdots & \vdots & \\
n & \varphi & E_{\exists}, m
\end{array}
$$

1. I would not be surprised if this leaves only Jeroen, Martin, and Roel de Vrijer as interested readers. The four of us tried to answer these questions around 1990, when *DPL* was being developed. But at some point we gave up because (i) things turned out to be more complicated than we thought, and (ii) we had more urgent things to do. I tried again in 1997, during a sabbatical spent in Edinburgh. I never published the result.

2. Throughout I will take the system of natural deduction presented in L.T.F. Gamut, *Logic, Language, and Meaning, volume 1, Introduction to Logic* as the starting point

With an appeal to this rule, one can easily show, for example, that in dynamic predicate logic

$$\forall x(Sx \to Px) \models \exists ySy \to Py.$$

Here is the derivation:

1	$\forall x(Sx \to Px)$	premise
2	$\exists ySy$	assumption
3	Sy	$\mathrm{E}_\exists$ 2
4	$Sy \to Py$	$\mathrm{E}_\forall$ 1
5	Py	$\mathrm{E}_\to$ 3,4
6	$\exists ySy \to Py$	$\mathrm{I}_\to$

It is important to realize that the sequence of formulas constituting this derivation is a *text* in the language of *DPL*. The variable y occurring in the formula on line 3 is bound by the quantifier $\exists y$ occurring in the formula on line 4, and so are all occurrences of y in line 4 and 5.

2 Some obstacles

Unfortunately, changing the elimination rule for the existential quantifier is not the only thing one has to do to get a system that covers the dynamic notion of validity. Here are some problems that one has to deal with.

(1) How would you go about proving that

$$Ac.\, Bc \models \exists xAx \wedge Bx$$

If you start like this :

1	Ac	premise
2	Bc	premise
3	$Ac \wedge Bc$	$\mathrm{I}_\wedge$ 1, 2
4	$\ldots$	

and next, at line 4, apply the classic introduction rule for $\exists$, you end up with $\exists x(Ax \wedge Bx)$, which is not what you want. And if you first apply the introduction rule for $\exists$, like this:

1	Ac	premise
2	Bc	premise
3	$\exists xAx$	$\mathrm{I}_\exists$ 1
4	$\ldots$	

then there seems to be no way to get Bx at line 4.

What is needed here is a generalization of of $\mathrm{I}_\exists$. It will be a rule that can be applied not only formulas but also to texts, as follows:

1	Ac	premise
2	Bc	premise
3	$\exists xAx$	
4	Bx	$\mathrm{I}_\exists$ 1-2
5	$\exists xAx \wedge Bx$	$\mathrm{I}_\wedge$ 3, 4

So, lines 3 and 4 are added in one go, by an application of $I_\exists$ to the text formed by the formulas on line 1 and line 2.

(2) Here is another problem. How would you go about showing that:

$$\exists x(Ax \wedge \exists x \neg Ax) \models \neg Ax$$

We cannot apply the new $E_\exists$ as described above here, because that would give:

1	$\exists x(Ax \wedge \exists x \neg Ax)$	premise
2	$Ax \wedge \exists x \neg Ax$	$E_\exists$
3	...	

This way, the variable x occurring in the first conjunct of the formula on line 2 gets bound by the second existential quantifier on line 1 —- with disastrous consequences. (We should impose conditions on $E_\exists$ that forbid this.)

The way out here is to introduce a new rule, *variable switch*, which enables us — under certain conditions — to replace a formula of the form $\exists x\varphi$ by a variant $\exists y[y/x]\varphi$. This is how things work out for the example at hand:

1	$\exists x(Ax \wedge \exists x \neg Ax)$	premise
2	$\exists y(Ay \wedge \exists x \neg Ax)$	variable switch, 1
3	$Ay \wedge \exists x \neg Ax$	$E_\exists$, 2
4	$\exists x \neg Ax$	$E_\wedge$, 3
5	$\neg Ax$	$E_\exists$, 4

(3) As an example of a third obstacle, note that:

$$\exists y Ay \wedge \exists x Bx, Cy \models \exists y Ay \wedge Cy$$

Whereas,

$$\exists y Ay \wedge \exists x Bx, Cy, \exists y Ay \not\models Cy$$

So, somehow the step 4 in the next derivation is invalid

1	$\exists y Ay \wedge \exists x Bx$	premise
2	Cy	premise
3	$\exists y Ay$	premise
4	Cy	repetition 2

whereas step 4 in the following derivation is fine.

1	$\exists y Ay \wedge \exists x Bx$	premise
2	Cy	premise
3	$\exists y Ay$	$E_\wedge$ 1
4	Cy	repetition 2
5	$\exists y Ay \wedge Cy$	$I_\wedge$, 3, 4

Intuitively, the difference is this. The formula $\exists y Ay$ on the third line in the first of these derivations adds a further premise to the premises already given. It introduces a new discourse referent, and all that is said about it is that the predicate A applies to it. So, to conclude that the predicate B applies to it, as is done in line 4 is invalid. The formula $\exists y Ay$ on the third line in the second derivation, on the other hand, is a conclusion drawn from the premises, and just recapitulates what was already said: There is an object with property A. And, yes, we already know about this object that it has property B as well.

It is not easy to characterize the difference formally. In the derivation we will have to keep track for each existential quantifier where it was first introduced, so that when it is just 'repeated' we can safely repeat things said about the referents introduced by it.

3 The system

Further details will be published on the website for this Festschrift.