QUANTUM LOGIC, QUANTUM HISTORIES AND QUANTUM TURING MACHINES

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1 Introduction

In his rich research Johan van Benthem has given some important contributions to *temporal* and *dynamic logics*. In this paper, we will present the main ideas of a semantic framework, where *quantum logic* permits us to model some typical temporal and dynamic situations. One is dealing with new *quantum logical structures* that have been suggested by the *consistent histories approach* to quantum theory (investigated by Gell-Mann, Hartle ([4]), Isham, Linden ([7], [8]) and many others).

Roughly, the basic idea of this approach is that the *fundamental objects* of quantum theory are represented by *consistent sets of histories*, where each *history* corresponds to a *possible temporal evolution* of a micro-physical system. Interestingly enough, one can find deep connections between *quantum histories* and *quantum computations*.

We propose a general semantics that, in principle, admits a number of possible applications. The basic concept in this semantics is the notion of *historical structure*.

2 Historical structures

Definition 2.1 A *historical structure* is structure

$$\mathcal{M} = \langle T,\, S,\, Ev,\, Histev\,, Op\,, \mathcal{D}\rangle\,,$$

where

1) T is a linearly ordered set of times. In the following a time-sequence $\langle t_1, \dots, t_n\rangle$ will be always written according to the order $\leq$.

2) S is a function that assigns to each $t \in T$ a set S_t of *possible states* at time t. For simplicity all S_t are supposed to be (ontologically distinct) copies of a fixed (timeless) S^*.

   ¿From an intuitive point of view, states can be regarded as *possible worlds* of a Kripke-style semantics: pieces of information about possible states of affairs (or fragments of reality).

   In the physical applications: S^* will (naturally) contain the *possible states* of the physical system under investigation. A state is called *pure* when it represents a *maximal information* that is consistent with respect to the theory. Non maximal pieces of information correspond to *mixed states*. In classical mechanics pure states are always *logically complete*. In other words any pure state can *semantically* decide any relevant physical property of the system (localization, velocity, energy,...). On the contrary, a characteristic aspect of quantum theory is a divergence between *maximality* and *logical completeness*. Owing to the *uncertainty relations*, any information about a quantum system is necessarily *logically incomplete*, in the sense that it cannot semantically decide *all* the physical properties concerning our system. In classical mechanics (CM), pure states are mathematically represented by *points* of an appropriate *phase space* Ω. In quantum theory (QT), instead, the pure states of a quantum object are represented by unitary vectors in an appropriate *Hilbert space* $\mathcal{H}$.

3) Ev is a function that assigns to each $t \in T$ a set $\mathcal{E}v_t$ of *events* at time t. For simplicity, all $\mathcal{E}v_t$ are supposed to be (ontologically distinct) copies of a fixed (timeless) $\mathcal{E}v^*$.

   In the case of our physical applications: $\mathcal{E}v^*$ will contain the possible physical properties of the physical system. In CM: measurable subsets of the phase space Ω. In QT: *closed subspaces* (or equivalently, *projection operators*) in the appropriate Hilbert space $\mathcal{H}$.

   Any timeless state s will assign to any timeless event α a value in the interval $[0, 1]$:
$$s(\alpha) \in [0, 1].$$

   A state s is said to *verify* an event α ($s \models \alpha$) iff $s(\alpha) = 1$. Similarly for all states in S_t and all events in $\mathcal{E}v_t$.

   The set $\mathcal{E}v^*$ of the timeless events has a structure (for simplicity all $\mathcal{E}v_t$ are assumed to be isomorphic to $\mathcal{E}v^*$). We will consider only examples of structures that are partially ordered by a relation $\sqsubseteq$.

   In the case of CM, $\mathcal{E}v^*$ gives rise to a *Boolean algebra* (a convenient subalgebra of the power-set of Ω). Similarly, in QT, the structure of $\mathcal{E}v^*$ can be identified with the algebra of all closed subspaces of $\mathcal{H}$ (a particular example of a *non distributive orthomodular lattice*). As a consequence one obtains that, differently from the classical mechanical case, quantum events do not have a classical logical behaviour.

   We will call *historical sequence* a sequence of events $\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle$, where each α_{t_i} is in $\mathcal{E}v_{t_i}$. Of course, the algebraic structure of each $\mathcal{E}v_{t_k}$ can be naturally transferred to the set of all historical sequences ($\prod_{i=1}^{n} \mathcal{E}v_{t_i}$). We will call *temporal support* ([7]) of the historical sequence $\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle$, the time-sequence $\langle t_1, \ldots, t_n \rangle$. Composition between temporal supports and historical sequences with disjoint temporal supports is defined in the expected way.

4) $Histev$ is a function that assigns to each time-sequence $\langle t_1, \ldots, t_n \rangle$ the set of the historical events $Histev^{\langle t_1, \ldots, t_n \rangle}$ at time $\langle t_1, \ldots, t_n \rangle$. This set is equipped with a structure.

   Physical examples:

   > In CM: $Histev^{\langle t_1, \ldots, t_n \rangle}$ is a σ-field of subsets of the *cartesian product* of
   > $$S_{t_1} \times \ldots \times S_{t_n}.$$
   > In other words, a historical event (at time $\langle t_1, \ldots, t_n \rangle$) is a set of sequences $\langle s_{t_1}, \ldots, s_{t_n} \rangle$, where any s_{t_i} is a pure state of the system at time t_i.

In QT: $Histev^{\langle t_1,\dots,t_n\rangle}$ can be identified with the set of the closed subspaces of the *tensor product*

$$\mathcal{H}_{t_1} \otimes \ldots \otimes \mathcal{H}_{t_n},$$

where each $\mathcal{H}_{t_i}$ represents the Hilbert space of the system at time t_i.

Any historical sequence of events must be represented by a historical event. However, not all historical events will represent historical sequences. For instance, in QT the sequence $\langle \alpha_{t_1},\dots,\alpha_{t_n}\rangle$ (where each α_{t_i} is a closed subspace) will be represented by the tensor product $\alpha_{t_1} \otimes \ldots \otimes \alpha_{t_n}$ of the closed subspaces $\alpha_{t_1},\dots,\alpha_{t_n}$. Such a product is a closed subspace in the tensor-product space $\mathcal{H}_{t_1} \otimes \ldots \otimes \mathcal{H}_{t_n}$. Of course, not all the closed subspaces in the tensor-product space will have this factorized form. For instance, the orthocomplement of a factorized historical event $\alpha_{t_1} \otimes \ldots \otimes \alpha_{t_n}$ (which represents the logical *negation* of the original event) will not generally correspond to any historical sequence of events. Our physical examples naturally suggest to require the following general conditions:

(4.1) For any time t_k and any time-sequence $\langle t_1,\dots,t_k,\dots,t_n\rangle$, there exists a function f that maps $\mathcal{E}v_{t_k}$ into $Histev^{\langle t_1,\dots,t_n\rangle}$:

$$f : \mathcal{E}v_{t_k} \to Histev^{\langle t_1,\dots,t_n\rangle}.$$

Further, such f is an embedding that preserves the algebraic structure of $\mathcal{E}v_{t_k}$.

This guarantees that any event at time t_k is represented by a historical event in any longer time-interval. For instance, in the case of closed subspaces, $f(\alpha_{t_k})$ will be $\mathbf{1}_{t_1} \otimes \ldots \otimes \alpha_{t_k} \otimes \ldots \otimes \mathbf{1}_{t_n}$, where $\mathbf{1}_{t_i}$ represents the *certain event* at time t_i.

(4.2) For any time sequence $\langle t_1,\dots,t_n\rangle$, there is a function g that transforms any historical sequence $\langle \alpha_{t_1},\dots,\alpha_{t_n}\rangle$ of events into a historical event $\eta^{\langle t_1,\dots,t_n\rangle}$ of $Histev^{\langle t_1,\dots,t_n\rangle}$:

$$g : \langle \alpha_{t_1},\dots,\alpha_{t_n}\rangle \mapsto \eta^{\langle t_1,\dots,t_n\rangle}.$$

Further, such g is a homomorphism of $\prod_{i=1}^{n} \mathcal{E}v_{t_i}$ into $Histev^{\langle t_1,\dots,t_n\rangle}$.

Definition 2.2 Two historical sequences $\langle \alpha_{t_1},\dots,\alpha_{t_n}\rangle$ and $\langle \beta_{t_1},\dots,\beta_{t_n}\rangle$ are called *equivalent* ($\langle \alpha_{t_1},\dots,\alpha_{t_n}\rangle \approx \langle \beta_{t_1},\dots,\beta_{t_n}\rangle$) iff $g(\langle \alpha_{t_1},\dots,\alpha_{t_n}\rangle) = g(\langle \beta_{t_1},\dots,\beta_{t_n}\rangle)$.

Since g is a homomorphism, the equivalence relation $\approx$ turns out to be a congruence on the algebraic structure induced on $\prod_{i=1}^{n} \mathcal{E}v_{t_i}$. Further, the relation $\approx$ is required to satisfy the following condition:

let $\langle t_m, \ldots, t_n \rangle$ and $\langle t_i, \ldots, t_j \rangle$ be two disjoint temporal supports: if $\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle \approx \langle \alpha'_{t_m}, \ldots, \alpha'_{t_n} \rangle$ and $\langle \beta_{t_i}, \ldots, \beta_{t_j} \rangle \approx \left\langle \beta'_{t_i}, \ldots, \beta'_{t_j} \right\rangle$, then $\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle \circ \langle \beta_{t_i}, \ldots, \beta_{t_j} \rangle \approx \langle \alpha'_{t_m}, \ldots, \alpha'_{t_n} \rangle \circ \left\langle \beta'_{t_i}, \ldots, \beta'_{t_j} \right\rangle$.

In other words, the relation $\approx$ is preserved under composition ($\circ$) of historical sequences.

Any historical event $\eta^{\langle t_1, \cdots, t_n \rangle}$ that represents (via g) a historical sequence will be called a *history* [1]. We will indicate by $Hist^{\langle t_1, \cdots, t_n \rangle}$ the subset of $Histev^{\langle t_1, \cdots, t_n \rangle}$ that contains all the histories. Instead of $g(\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle)$ we will simply write: $\eta^{\left\langle \alpha_{t_1}, \cdots, \alpha_{t_n} \right\rangle}$.

On this basis one can naturally define a notion of *historical truth*. This is a semantic notion that may hold between a sequence of states and a historical event. Let us first consider the case where both the state-sequence and the historical event refer to the same temporal support $\langle t_1, \ldots, t_n \rangle$.

Definition 2.3 *Restricted definition of historical truth*
($\langle s_{t_1}, \ldots, s_{t_n} \rangle$ *verifies* η: $\langle s_{t_1}, \ldots, s_{t_n} \rangle \models \eta$)

We will distinguish the case of histories from that of historical events that do not represent histories.

a) Let η be the history $\eta^{\left\langle \alpha_{t_1}, \cdots, \alpha_{t_n} \right\rangle}$. Then:
   $\langle s_{t_1}, \ldots, s_{t_n} \rangle \models \eta$ iff for any sequence $\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle$ s.t. $g(\langle \alpha_{t_1}, \ldots, \alpha_{t_n} \rangle) = \eta$: $s_{t_i} \models \alpha_{t_i}$, for any s_{t_i} and α_{t_i} $(1 \leq i \leq n)$.
b) Let η be a historical event (belonging to $Histev^{\langle t_1, \cdots, t_n \rangle}$) that is not a history. Then:
   $\langle s_{t_1}, \ldots, s_{t_n} \rangle \models \eta$ iff for at least one history $\delta^{\left\langle \alpha_{t_1}, \cdots, \alpha_{t_n} \right\rangle}$:
   i) $\delta^{\left\langle \alpha_{t_1}, \cdots, \alpha_{t_n} \right\rangle} \sqsubseteq \eta$, where $\sqsubseteq$ is the partial order of the event structure;
   ii) $\langle s_{t_1}, \ldots, s_{t_n} \rangle \models \delta^{\left\langle \alpha_{t_1}, \cdots, \alpha_{t_n} \right\rangle}$.

Our truth definition can be naturally extended also to the case where a state-sequence and a historical event refer to different time-sequences. Let us first introduce a procedure that permits us to normalize any history to a given time-sequence.

Definition 2.4 *Normalization of a historical-sequence to a given time sequence*

Let $\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle$ be a historical sequence and let $\langle t_i, \ldots, t_j \rangle$ be any time-sequence. The *normalization* of $\langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle$ to $\langle t_i, \ldots, t_j \rangle$ is the following historical sequence:

$$\lceil \langle \alpha_{t_m}, \ldots, \alpha_{t_n} \rangle \rceil^{\langle t_i, \cdots, t_j \rangle} := \left\langle \beta_{t_i}, \ldots, \beta_{t_j} \right\rangle,$$

[1] In [7], historical sequences are called *history-filters*, while historical events are called *history -propositions*.

where

$$\beta_{t_k} = \begin{cases} \alpha_{t_k}, & \text{if } t_k \text{ is in } \langle t_m, \dots, t_n \rangle; \\ \mathbf{1}_{t_k} & \text{otherwise.} \end{cases}$$

($\mathbf{1}_{t_k}$ is the certain event at time t_k).

Accordingly, the normalization of a history will be automatically determined (via g):

Definition 2.5 *Normalization of a history to a given time-sequence*
Let $\eta^{\langle \alpha_{t_m}, \dots, \alpha_{t_n} \rangle}$ be a history and $\langle t_i, \dots, t_j \rangle$ be a time-sequence. The *normalization* of $\eta^{\langle \alpha_{t_m}, \dots, \alpha_{t_n} \rangle}$ to $\langle t_i, \dots, t_j \rangle$ is the history

$$\lceil \eta^{\langle \alpha_{t_m}, \dots, \alpha_{t_n} \rangle} \rceil^{\langle t_i, \dots, t_j \rangle}$$

that is univocally determined by the historical sequence

$$\lceil \langle \alpha_{t_m}, \dots, \alpha_{t_n} \rangle \rceil^{\langle t_i, \dots, t_j \rangle}.$$

One can easily prove that Definition 2.5 is a good definition since it is independent of the choice of the representative.

On this basis we can define a general notion of historical truth:

Definition 2.6 *General definition of historical truth*
($\langle s_{t_1}, \dots, s_{t_n} \rangle$ *verifies* $\eta^{\langle t_i, \dots, t_j \rangle}$: $\langle s_{t_1}, \dots, s_{t_n} \rangle \models \eta^{\langle t_{t_i}, \dots, t_{t_j} \rangle}$)
Let $\eta^{\langle t_i, \dots, t_j \rangle}$ be a historical event.
$\langle s_{t_1}, \dots, s_{t_n} \rangle \models \eta^{\langle t_i, \dots, t_j \rangle}$ iff there is a history $\delta^{\langle t_i, \dots, t_j \rangle}$ s.t.:

i) $\delta^{\langle t_i, \dots, t_j \rangle} \sqsubseteq \eta^{\langle t_i, \dots, t_j \rangle}$;
ii) $\langle s_{t_1}, \dots, s_{t_n} \rangle \models \lceil \delta^{\langle t_i, \dots, t_j \rangle} \rceil^{\langle t_1, \dots, t_n \rangle}$.

A historical sequence will be called *normal* when all historical events can be normalized to any time sequence.

It turns out that any historical structure where all $Histev^{\langle t_1, \dots, t_n \rangle}$ are structured as complete lattices is trivially normal.

5) Op, the *operation function*, is a function that associates to any pair of time t_i, t_j a set of *operations* $\mathcal{O}_{t_i}^{t_j}$

$$Op : \langle t_i, t_j \rangle \mapsto \mathcal{O}_{t_i}^{t_j},$$

where $\mathcal{O}_{t_i}^{t_j}$ is a subset of a set of admissible operations $\mathcal{O} \subseteq \left\{ f \mid \overline{S^*} \to S^* \right\}$ ($\overline{S^*} \subseteq S^*$). In other words, admissible operations transform states into states.

Some operations in $\mathcal{O}_{t_i}^{t_j}$ may represent spontaneous evolutions whereas other operations may represent state transformations induced by an action. A typical action is a test performed in order to check whether a certain property α holds[2]. Of course, physical measurements are paradigmatic examples of action of this kind. We will indicate by $f^{\alpha}(s)$ the transformation of a state s induced by an action that has been performed in order to check whether the object in state s satisfies the property α.

On this basis a class of *accessibility relations* can be naturally defined in terms of our operations (differently from Kripke semantics, where accessibility relations are usually dealt with as primitive). Let s, u represent timeless states.

States s and u are called *accessible in the time interval* $[t_i, t_j]$ (we will write $Acc_{t_i}^{t_j}(s, u)$) iff there exists an operation $f \in \mathcal{O}_{t_i}^{t_j}$ s.t.

$$u = f(s).$$

States s and u are *absolutely accessible* ($Acc(s, u)$) iff for at least two times t_i, t_j:

$$Acc_{t_i}^{t_j}(s, u).$$

Let α be an event. The states s, u are *α-accessible* ($Acc^{\alpha}(s, u)$) iff there exists an operation f^{α} s.t.

$$u = f^{\alpha}(s).$$

Since all S_t are copies of S^* our accessibility relations are automatically transferred to pairs of states that may belong to different S_t.

6) $\mathcal{D}$ is a (possibly empty) set of *decoherence functionals* d ([7]). From the intuitive point of view, $d(\eta, \delta)$ measures the degree of interference between the historical events η and δ.

3 Dynamic and temporal operators

Different logical operators that have a temporal or a dynamic meaning can be naturally defined in our semantics.

Definition 3.1 *The temporal conjunction* **and then** $\sqcap\!\!\nearrow$
Let $\eta^{\langle t_m,\cdots,t_n\rangle}, \delta^{\langle t_i,\cdots,t_j\rangle}$ be two historical events. Generally $\sqcap\!\!\nearrow$ is a partial operation that turns out to be always defined in the particular case of a normal historical structure where any $Histev^{\langle t_1,\cdots,t_n\rangle}$ gives rise to a lattice (where $\sqcap$ and $\sqcup$ represent the *infimum* and the *supremum*, respectively).

$$\eta^{\langle t_m,\cdots,t_n\rangle} \sqcap\!\!\nearrow \delta^{\langle t_i,\cdots,t_j\rangle} := \begin{cases} \mathbf{0}^{\langle t_m,\cdots,t_n\rangle\circ\langle t_i,\cdots,t_j\rangle}, \text{ if } \langle t_m,\ldots,t_n\rangle \text{ does not preceed} \\ \qquad\qquad \langle t_i,\ldots,t_j\rangle \text{ w.r.t the order } \leq; \\ \lceil\eta\rceil^{\langle t_m,\cdots,t_n\rangle\circ\langle t_i,\cdots,t_j\rangle} \sqcap \lceil\delta\rceil^{\langle t_m,\cdots,t_n\rangle\circ\langle t_i,\cdots,t_j\rangle}, \text{ if this inf} \\ \qquad\qquad \text{exists in } Histev^{\langle t_m,\cdots,t_n\rangle\circ\langle t_i,\cdots,t_j\rangle}; \\ \text{undefined}, \qquad \text{otherwise.} \end{cases}$$

[2]See von Wright [14].

As an example, suppose two histories $\eta^{\langle \alpha_{t_1}, \alpha_{t_2} \rangle}$, $\delta^{\langle \beta_{t_3}, \beta_{t_4} \rangle}$ $(t_1 < t_2 < t_3 < t_4)$. According to our definition we will have:

$$\begin{aligned}
\eta^{\langle \alpha_{t_1}, \alpha_{t_2} \rangle} \sqcap\!\!\nearrow \delta^{\langle \beta_{t_3}, \beta_{t_4} \rangle} &= \eta^{\langle \alpha_{t_1}, \alpha_{t_2}, \mathbf{1}_{t_3}, \mathbf{1}_{t_4} \rangle} \sqcap \delta^{\langle \mathbf{1}_{t_1}, \mathbf{1}_{t_2}, \beta_{t_3}, \beta_{t_4} \rangle} \\
&= \gamma^{\langle \alpha_{t_1} \sqcap \mathbf{1}_{t_1}, \alpha_{t_2} \sqcap \mathbf{1}_{t_2}, \mathbf{1}_{t_3} \sqcap \beta_{t_3}, \mathbf{1}_{t_4} \sqcap \beta_{t_4} \rangle} \\
&= \gamma^{\langle \alpha_{t_1}, \alpha_{t_2}, \beta_{t_3}, \beta_{t_4} \rangle}.
\end{aligned}$$

We obtain in this way the expected meaning of a temporal conjunction: $\eta^{\langle \alpha_{t_1}, \alpha_{t_2} \rangle}$ *and then* $\delta^{\langle \beta_{t_3}, \beta_{t_4} \rangle}$ is the history determined by the historical sequence $\langle \alpha_{t_1}, \alpha_{t_2}, \beta_{t_3}, \beta_{t_4} \rangle$. **A dynamic implication**

Let us refer to set $\mathcal{E}v^*$ of all timeless events and to the set S^* of all timeless states. Let $\alpha \in \mathcal{E}v^*$. We define first a function $\boxed{\boxed{\alpha}}$ that assign to each $\beta \in Ev^*$ a set of timeless states:

$$\boxed{\boxed{\alpha}} : \mathcal{E}v^* \to \mathcal{P}(S^*) \qquad \text{(where } \mathcal{P}(S^*) \text{ is the power-set of } S^*\text{)}$$

satisfying the condition:

$$s \in \boxed{\boxed{\alpha}}\beta \quad \text{iff} \quad \forall u \, [Acc^{\alpha}(s,u) \Rightarrow u \models \beta \,] \, .$$

¿From the intuitive point of view, $s \in \boxed{\boxed{\alpha}}\beta$ has the following meaning: suppose we test α on a state s and we obtain a positive result; then, any state into which s is transformed after such a test verifies β. Of course, $\boxed{\boxed{\alpha}}\beta$ does not necessarily determine an event. In the case where $\mathcal{E}v^*$ has the structure of a complete lattice, one can easily define the following total operation:

$$\boxed{\alpha} : \mathcal{E}v^* \to \mathcal{E}v^*,$$

where

$$\boxed{\alpha}\beta := \bigsqcap \left\{ \delta \mid \forall u \in \boxed{\boxed{\alpha}}\beta : \, u \models \delta \right\}$$

Otherwise, $\boxed{\alpha}$ will represent a partial operation.

In the particular case of the lattice of all closed subspaces in a Hilbert space $\mathcal{H}$ the dynamic implication is always defined and corresponds to the usual quantum logical (material) implication (which is also called *Sasaki implication*).

As is well known, differently from classical logic, $\alpha' \sqcup \beta$ (where $'$ is the *orthocomplement*) does not represent a *good conditional operation* in quantum logic. Actually, it may happen that in a given orthomodular lattice, $\alpha' \sqcup \beta$ is equal to $\mathbf{1}$, even if α does not preceed β according to the lattice-order. From the logical point of view, this means that a sentence like "*not A or B*" might be true in a given (algebraic) model, even if A does not imply B in the same model.

However, a particular variant of $\alpha' \sqcup \beta$ (which is equivalent to $\alpha' \sqcup \beta$ in all Boolean lattices) permits us to define a *good quantum logical conditional*. It is sufficient to put:

$$\alpha \to \beta := \alpha' \sqcup (\alpha \sqcap \beta).$$

One can easily show that for closed subspaces α, β:

$$\boxed{\alpha}\beta = \alpha \rightarrow \beta.$$

Of course, such a relation holds, only if we interpret the accessibility relation Acc^{α} in the "natural" quantum theoretical way. In other words:

$Acc^{\alpha}(s, u)$ iff u is the result of the projection of s over the closed subspace α.

This shows that the standard logical implication admits a natural dynamic interpretation. In other words, even the orthodox version of quantum logic (which is usually described as a "static logic") seems to have some "hidden" dynamic features.

4 Quantum histories and quantum Turing machines

Let us first introduce a definition of *quantum Turing machine*, following a quite elegant abstract approach, that has been recently proposed by Gudder ([5]).

Definition 4.1 A *quantum Turing machine*, whose *tape* is identified with the set of the integers $\mathcal{Z}$, is a structure

$$M = \langle I, S, \delta, \mathcal{H} \rangle$$

where:

1) I is a *finite alphabet*, containing a blank symbol $\sharp$.

2) S is a set of *memory states*, containing an initial state s_0 and a final state s_f. Similarly to classical Turing machines, consider the cartesian product

$$S \times I \times I \times \{L, R\} \times S.$$

The interpretation of any element $\langle s, x, y, d, r \rangle$ (where $d \in \{R, L\}$) of our product will be the following: the machine M *is* in state s, *sees* the symbol x, *prints* the symbol y, *goes* to d and finally *transits* to state r. Any $\langle s, x, y, d, r \rangle$ represents a *computational event*.

3) δ is the *amplitude function* that assigns to any computational event $\langle s, x, y, d, r \rangle$ a complex number. The number $\mid \delta(\langle s, x, y, d, r \rangle) \mid^2$ will represent the *probability* of the computational event $\langle s, x, y, d, r \rangle$.

M evolves in time assuming different states. A *configuration* (or *basic state*) of M is a triplet:

$$\varphi = \langle n, s, w \rangle$$

where n is a *location* of M on the tape, s is a memory state and w is a *word*, describing the *condition of the tape*. Similarly to the classical case, a word w can be described as function

$$w : \mathcal{Z} \rightarrow I$$

with the usual restriction: for almost all $n \in \mathcal{Z}$: $w(n) = \sharp$. Hence, a word corresponds to a finite sequence of symbols labelled by the cells of the tape. Instead of $w(n)$ we will also write w_n.

Let B be the set of all possible basic states φ (which describe definite configurations of M).

4) $\mathcal{H}$ is a Hilbert space with *orthonormal basis* B.

The amplitude function δ determines a linear operator

$$U : \mathcal{H} \to \mathcal{H}.$$

It is sufficient to define U on B:

for any $\varphi \in B$, $U(\varphi)$ will generally be a superposition state:

$$\sum_i c_i \varphi_i.$$

Let $\varphi = \langle n, s, w \rangle$ and let w_n represent the symbol printed in the cell n.

Let us consider the set of all computational events starting with the pair $\langle s, w_n \rangle$ (which is determined by φ):

$$\langle s, w_n, y, d, r \rangle$$

for all possible y, d, r.

For any y, d, r, we define the complex number $c_{y,d,r}$ and the vector $\varphi_{y,d,r}$ as follows:

$$c_{y,d,r} = \delta(\langle s, w_n, y, d, r \rangle)$$

$$\varphi_{y,d,r} = \langle n(d), r, w(y, n) \rangle .$$

Where:

$$n(d) = \begin{cases} n + 1, & \text{if } d = R; \\ n - 1, & \text{if } d = L, \end{cases}$$

and:

$$w(y, n)_m = \begin{cases} y, & \text{if } m = n; \\ w_m, & \text{otherwise.} \end{cases}$$

Now, let us define:

$$U(\varphi) = \sum_{y,d,r} c_{y,d,r} \varphi_{y,d,r}.$$

We assume the following restriction: U must be a *unitary* operator. Obviously, such a restriction represents a constraint on the amplitude function δ.

How can the *action* of our unitary operator U be interpreted? Let $\psi_0 = \langle 0, s_0, w \rangle$ represent the element of B that is the initial state, where the word w corresponds to the input for M. The state $U(\psi_0)$ will represent the evolution of the initial state ψ_0 after one step. Generally $U^t(\psi_0)$ will represent the evolution of ψ_0 after t steps (we might also say: at time t).

What does it mean that our machine M halts at time t with a certain probability value? In order to answer this question, let us first introduce the notion of *final basic state*.

Definition 4.2 A *final basic state* is a vector $\varphi \in B$ s.t.:

$$\varphi = \langle n, s_f, w \rangle ,$$

where s_f is the final memory state.

We will denote by B^F set of all final basic states.

Definition 4.3 The *probability that M halts at time t* ($Prob^H(t)$) is defined as follows:

$$Prob^H(t) = \sum_{\varphi \in B^F} \mid (U^t(\psi_0), \varphi) \mid^2$$

(where $(U^t(\psi_0), \varphi)$ represents the *inner product* of $U^t(\psi_0)$ and φ).

¿From an intuitive point of view, any sequence

$$\left\langle \psi_0, ..., U^t(\psi_0) \right\rangle$$

can be regarded as a *knowledge path* (or an *epistemic history*) of M. Such a sequence naturally gives rise to a graph, where each branch has the form:

$$\langle \psi_0, \varphi_1, ..., \varphi_t \rangle$$

with $\varphi_i \in B$.

These basic sequences $\langle \psi_0, \varphi_1, ..., \varphi_t \rangle$ are determined as follows. For any k ($0 \leq k \leq t$), consider

$$U^k(\psi_0) = \sum_i c_i \varphi_i \quad \text{(with } c_i \neq 0\text{)}.$$

We require:

$$\varphi_k = \varphi_i, \quad \text{for a given } \varphi_i \text{ occurring in } \sum_i c_i \varphi_i .$$

Since states correspond to unidimensional closed subspaces of $\mathcal{H}$, any knowledge path of M represents a *historical sequence* (in the sense of Definition 2.1).

On this basis, one can distinguish at least two different notions of *computation*.

Definition 4.4 A *strong computation* of M is a knowledge path

$$\left\langle \psi_0, ..., U^t(\psi_0)\right\rangle$$

where M halts at time t with probability 1.

Definition 4.5 A *weak computation* is a knowledge path

$$\left\langle \psi_0, ..., U^t(\psi_0)\right\rangle$$

where M halts at t with probability different from 0.

In this framework, one can naturally introduce some semantic notions (following a standard quantum logical style).

Definition 4.6 A basic state φ *knows* (or *accepts*) a word w ($\varphi \models w$) iff there is a location n and a memory state s such that

$$\varphi = \langle n, s, w \rangle$$

Definition 4.7 A state ψ *possibly knows* (or *possibly accepts*) a word w ($\psi \ \Vdash w$) iff $\psi = \sum c_i \varphi_i$ with $c_i \neq 0$ and for at least one φ_i,

$$\varphi_i \models w.$$

Definition 4.8 M *possibly knows* a word w at time t ($M \Vdash_t w$) iff

$$U^t(\psi_0) \Vdash w.$$

Definition 4.9 M *knows* a word w at time t ($M \models_t w$) iff

1) $M \Vdash_t w$;

2) M halts at time t with probability 1.

Definition 4.10 The *proposition* $\mathcal{C}(w)$ of a word w is the smallest closed subspace that includes

$$\{\varphi \in B \mid \varphi \models w\}.$$

Suppose we equip our language with the logical connectives $\neg$ and $\wedge$ (where the atomic formulas can be identified with particular words). Then we can give the usual (quantum logical) semantic definitions:

Definition 4.11

$\psi \Vdash \neg\alpha$ iff $\psi \in \mathcal{C}(\alpha)'$;

$\psi \Vdash \alpha \wedge \beta$ iff $\psi \in \mathcal{C}(\alpha) \cap \mathcal{C}(\beta)$.

Suppose $M \Vdash_t \alpha$ and $M \models_t \alpha$ are defined as above.

It may happen:

$$\exists t : M \models_t \neg\alpha$$

And

$$\exists t : M \models_t \alpha$$

In other words, the negation $\neg$ shows in this framework a *weak paraconsistent* behaviour (similarly to a *negation by failure*).

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