Indiscernibility and complementarity relations in information systems

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Abstract

We present logical systems for an analysis of data that have the form of descriptions of some objects of an application domain in terms of their attributes. We analyse two types of relationships among objects referred to as indiscernibility and complementarity. We present a modal logic LIC for reasoning about indiscernibility, complementarity and relationships between them. We define a Kripke-style semantics for LIC as well as semantics determined by information systems. We present a sound and complete deduction system for LIC. We also investigate the complexity of the satisfiability problem for LIC.

Contents

1 Introduction

We present logical systems for an analysis of data that have the form of descriptions of some objects of an application domain in terms of their attributes. We analyse two types of relationships among objects referred to as indiscernibility and complementarity. Each of them separetely has been extensively studied in the literature (e.g. [Vak90, DO97, DO99]), but until now no formal attempt has been made toward a theory of interconnections between indiscernibility and complementarity. In this paper we concentrate on developing logical tools for dealing with problems that involve both indiscernibility and complementarity. Intuitively, the objects having the same description are indiscernible and the objects whose descriptions are complements of each other are complementary. We present a modal logic LIC for reasoning about indiscernibility, complementarity and relationships between them. We define a Kripke-style semantics for LIC as well as semantics determined by information systems. We present a sound and complete deduction system for LIC. We also investigate the complexity of the satisfiability problem for LIC. We briefly mention applications that require the computation of both indiscernibility and complementarity.

We dedicate this paper to Johan van Benthem. His important results on modal logics, their methodology and applications have been a source of inspiration for us.

2 Information systems

For algorithmic reasons information about objects is often presented in the form of a table. The rows of the table are labelled with objects, the columns are labelled with attributes, and the entries of the table are collections of sets of values of attributes. Formally, an *information system* is a structure of the form

$$\langle OB, AT, \{VAL_a : a \in AT\}, f\rangle$$

where OB is a nonempty set of *objects*, AT is a nonempty set of *attributes*, VAL_a is a nonempty set of *values* of the attribute a and f is a total function $OB \times AT \rightarrow \mathcal{P}(\bigcup_{a \in AT} Val_a)$ such that for any $\langle x, a\rangle \in OB \times AT$, $f(x,a) \subseteq Val_a$. We shall also use the more concise notation $\langle OB, AT\rangle$ for $\langle OB, AT, \{VAL_a : a \in AT\}, f\rangle$. An information system $\langle OB, AT\rangle$ is *total* [resp. *deterministic*] $\stackrel{\text{def}}{\Leftrightarrow}$ for any $a \in AT$ and for any $x \in OB$, $f(x,a) \neq \emptyset$ [resp. $card(f(x,a)) \leq 1$]. Consider a simple example [DO98]:

	colour	size	d
x_1	green	small	yes
x_2	not green	small	yes
x_3	blue	medium	no

The notion of a set-theoretical information system introduced in [Vak87] plays an important role in the developments of this paper. Let $\langle W, V\rangle$ be a pair such that W is a non-empty set and V included in $\mathcal{P}(\mathcal{P}(W))$ is a non-empty set of families of subsets of W such that all elements of V are non-empty. We define the *set-theoretical information system* $\mathcal{S} = \langle OB, AT \rangle$ under $\langle W, V \rangle$ as follows:

- $OB \stackrel{\text{def}}{=} W$;
- $AT \stackrel{\text{def}}{=} \{a^X : X \in V\}$ such that $a^X : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ and for any $x \in OB$ $a^X(x) \stackrel{\text{def}}{=} \{Y : Y \in V,\ x \in Y\}$. So $VAL_{a^X} = X$ for $X \in V$;
- for any $x \in OB$ and for any $a^X \in AT$, $f(x, a^X) \stackrel{\text{def}}{=} a^X(x)$.

Any set-theoretical information system $\langle OB, AT \rangle$ under $\langle W, V \rangle$ is sometime abusively denoted by $\langle W, V \rangle$ itself.

3 Indiscernibility and complementarity derived from an information system

Let $\mathcal{S} = \langle OB, AT \rangle$ be an information system and A be a subset of AT. We define families of indiscernibility relations $\equiv_A$ and complementarity relations R_A on the set OB as follows: for all objects $x, y \in OB$

- $x \equiv_A y \stackrel{\text{def}}{\Leftrightarrow}$ for all $a \in A$, $f(x, a) = f(y, a)$;
- $xR_Ay \stackrel{\text{def}}{\Leftrightarrow}$ for all $a \in A$, $f(x, a) = VAL_a \setminus f(y, a)$.

If $A = \{a\}$ is a singleton, we write $\equiv_a$ and R_a, respectively. If $A = AT$, we often write $\equiv_{\mathcal{S}}$ and $R_{\mathcal{S}}$, respectively. Sometimes, when this will not cause a confusion, we will omit the subscript $\mathcal{S}$ in the above relations.

Lemma 3.1. Let $\mathcal{S} = \langle W, V \rangle$ be a set-theoretical information system.

(I) $x \equiv_{\mathcal{S}} y$ iff for all $a \in V$, for all $v \in a$, $x \in v$ iff $y \in v$;

(II) $xR_{\mathcal{S}}y$ iff for all $a \in V$, for all $v \in a$, $x \in v$ iff $y \notin v$;

(III) $\mathcal{S}$ is deterministic iff for all $x \in W$, for all $a \in V$, for all $v, v' \in a$, $x \in (v \cap v')$ implies $v = v'$.

As usual, for any relation S on W and for $x \in W$, we define $S(x) \stackrel{\text{def}}{=} \{y \in W : xSy\}$.

4 Frames with indiscernibility and complementarity

An *IC-frame* (Indiscernibility + Complementarity) is a relational system $\mathcal{F} = \langle W, \equiv, R \rangle$ such that W is a nonempty set and $\equiv$ and R are binary relations on W satisfying the following conditions:

(S1) $x \equiv x$;

(S2) $x \equiv y$ implies $y \equiv x$;

(S3) $x \equiv y$ and $y \equiv z$ imply $x \equiv z$;

(S4) xRy implies yRx;

(S5) xRy and $y \equiv z$ imply xRz;

(S6) xRy and yRz imply $x \equiv z$;

(S7) not xRx.

In any IC-frame $\langle W, \equiv, R\rangle$, R is a complementarity relation in the sense of the definition given in [DO98], that is R is irreflexive, symmetric and 3-transitive. Given an information system $\mathcal{S} = \langle OB, AT\rangle$, the relations $\equiv_{\mathcal{S}}$ and $R_{\mathcal{S}}$ satisfy the conditions (S1)-(S7). Hence, the relational system $\langle OB, \equiv_{\mathcal{S}}, R_{\mathcal{S}}\rangle$ is an IC-frame.

Any IC-frame with the relations derived from an information system in the above way is referred to as a *standard IC-frame*. In sections 7 and 8 the following classes of IC-frames will be investigated. Let $\mathcal{K}$, $\mathcal{KS}$, $\mathcal{KSD}$, $\mathcal{KST}$ be the class of all the IC-frames, standard IC-frames, standard IC-frames derived from a deterministic information system, and standard IC-frames derived from a total information system, respectively. Furthermore, let $\mathcal{KNS}$ be the class of relational systems that satisfy conditions (S1),...,(S6). They are referred to as *non-standard IC-frames*.

5 Applications of indiscernibility and complementarity

Many applications of indiscernibility relations alone are known in the literature, see for example [BO97, DO97, Orło83, Orło90, OP84]. In this section we briefly describe applications which require both indiscernibility and complementarity. An information system $\mathcal{S} = \langle OB, AT\rangle$ such that the set of attributes is finite and is partitioned into two subsets $AT = C \cup D$, $C \cap D = \emptyset$, C and D are the set of condition attributes and the set of decision attributes, respectively, is referred to as decision table. Consider the information system from Section 2 and assume that $C = \{colour, size\}$ and $D = \{d\}$.

¿From a decision table a set of decision rules can be derived relating descriptions of objects in terms of conditions to decisions about these objects. Let $C = \{c_1, ..., c_n\}$. For any object $x \in OB$, the rule r_x derived from $\mathcal{S}$ has the form:

> r_x: if the value of c_1 for y is $f(x, c_1)$ and ...and the value of c_n for y is $f(x, c_n)$ then $y \in Y$, where $Y = \equiv_D (x)$.

In our example, the equivalence classes of $\equiv_d$ are $Y_1 = \{x_1, x_2\}$ and $Y_2 = \{x_3\}$. The decision rule determined by an object x_1 is:

> r_{x_1}: if an object is green and small then it belongs to class Y_1.

An object o supports a decision rule r_x if its description matches both the condition part and the decision part of the rule, that is if $o \in \equiv_{AT} (x)$. An important problem in designing algorithms for synthesis of decision rules is to find heuristics for making the rules sufficiently general, so that they will be supported by many objects. Reduction of attributes and reduction of values of attributes are often parts of these heuristics. Computation of indiscernibility and complementarity provides a tool for verification of the conditions of reducibility of the respective data. A condition attribute a is redundant in the rule r_x if there is an $y \in OB$ such that:

1. $xR_a y$;
2. $x \equiv_{C-\{a\}} y$;
3. for every $b \in C - \{a\}$, $\equiv_b (x) \subseteq \equiv_D (x)$.

Intuitively, an attribute a is redundant in a rule if the decision made on the basis of that rule does not depend on a. Continuing our example, we can easily see that attribute *colour* is redundant in the rule r_{x_1}. The reduced rule is supported by x_1 and x_2.

The process of reduction of values of attributes is referred to as *contraction of attributes*. An attribute b is a contraction of attribute a if there is a mapping $\alpha : VAL_a \rightarrow VAL_b$ such that for every $x \in OB$, $f(x, a) = \alpha(f(x, b))$. Of special importance is a contraction to features. An attribute a is referred to as a feature if $VAL_a = \{0, 1\}$. It is known that for every attribute a there is a set $F(a)$ of features such that $\equiv_a = \equiv_{F(a)}$ [Iwi88]. It is easy to see that features can be characterised in terms of indiscernibility and complementarity.

LEMMA 5.1. Let $\mathcal{S} = \langle OB, AT \rangle$ be a deterministic information system.

(I) If an attribute a is a feature then
(*) for every $x, y \in OB$ either $x \equiv_a y$ or $xR_a y$;

(II) If (*) holds then there is a feature b such that $\equiv_a = \equiv_b$.

Consider the following decision table below left:

	colour	d
x_1	light green	+
x_2	red	-
x_3	blue	-
x_4	dark green	+

	c	d
x_1	green	+
x_2	red	-
x_3	blue	-
x_4	green	+

Suppose that we are not interested in the tints of colours, then we can contract the attribute colour and replace the first table by the table above right. This decision table leads to a smaller number of rules supported by more objects. If furthermore, we contract c to the attribute with values: green for x_1 and x_4 and not green for x_2 and x_3, then we obtain two decision rules that are in agreement with the original data. Of course the real situations are much more involved and various other constraints have to be taken into account. Our only aim here is to give an intuitive account of the role of indiscernibility and complementarity.

6 First-order characterization of IC-frames

In this section we show that every IC-frame is a standard IC-frame that is derived from a deterministic information system. Let $\mathcal{F} = \langle W, \equiv, R \rangle$ be an IC-frame. A subset $X \subseteq W$ is called an $\equiv$-set $\stackrel{\text{def}}{\Leftrightarrow}$ for any $x, y \in W$, if $x \in X$ and $x \equiv y$, then $y \in X$. Equivalently, X is an union of equivalence classes of $\equiv$. A set X is called an R-set $\stackrel{\text{def}}{\Leftrightarrow}$ it is a $\equiv$-set and for any $x, y \in X$, we have not xRy. X is called a *good set* $\stackrel{\text{def}}{\Leftrightarrow}$ it is an R-set and its complement $W \setminus X$ is also an R-set. Obviously, $\emptyset$ is an R-set.

Lemma 6.1. Let $\mathcal{F} = \langle W, \equiv, R\rangle$ be an IC-frame.

(I) $\equiv(x)$ is the smallest $\equiv$-set containing x;

(II) The set of all $\equiv$-sets is closed under the operations of complementation (wrt W), arbitrary unions and intersections;

(III) if not xRy, then $\equiv(x) \cup \equiv(y)$ is an R-set;

(IV) for any $x \in W$, $\equiv(x)$ is an R-set;

(V) if $(X_i)_{i\in I}$ is a chain of R-sets, then $\bigcup_{i\in I} X_i$ is an R-set;

(VI) If X is an $\equiv$-set, then $X \cup \equiv(y)$ is the smallest $\equiv$-set containing X and y.

(VII) If X is an R-set then $X \cup \equiv(y)$ is an R-set iff for any $x \in X$, not xRy.

The proof is by an easy verification.

Lemma 6.2. Let $\mathcal{F} = \langle W, \equiv, R\rangle$ be an IC-frame and X and Y be R-sets such that $X \cap Y = \emptyset$. Then for any $x \in W$ one of the following conditions is satisfied:

(I) $X \cup \equiv(x)$ is an R-set and $(X \cup \equiv(x)) \cap Y = \emptyset$;

(II) $Y \cup \equiv(x)$ is an R-set and $X \cap (\equiv(x) \cup Y) = \emptyset$;

Proof: Suppose that neither (I) nor (II) holds. ¿From not (I) and Lemma 6.1(VII), (i) there is $x_1 \in X$ such that $x_1 R x$ or (ii) there is $y_1 \in Y$ such that $x \equiv y_1$ (remember $X \cap Y = \emptyset$). Similarly, from not (II) and Lemma 6.1(VII), (iii) there is $x_2 \in Y$ such that $x_2 R x$ or (iv) there is $y_2 \in X$ such that $x \equiv y_2$ (remember $X \cap Y = \emptyset$). Four cases can be distinguised.
(i) and (iii): By (S4) and (S6), we obtain $x_1 \equiv x_2$. So $x_2 \in X$, a contradiction since $X \cap Y = \emptyset$.
(i) and (iv): By (S5), $x_1 R y_2$, a contradiction since for all $z, z' \in X$, not zRz'.
(ii) and (iii): By (S5), $x_2 R y_1$, a contradiction since for all $z, z' \in X$, not zRz'.
(ii) and (iv): Since $\equiv$ is an equivalence relation, $y_1 \equiv y_2$. Since X and Y are R-sets, $\{y_1, y_2\} \subseteq X \cap Y$, a contradiction. **Q.E.D.**

Theorem 6.3. (Separation theorem for R-sets) Let X and Y be R-sets such that $X \cap Y = \emptyset$. Then there exists a good set Z such that $X \subseteq Z$ and $Y \subseteq (W \setminus Z)$.

Proof: Let M be the set of R-sets X' such that $X \subseteq X'$ and $X' \cap Y = \emptyset$. M is non-empty since $X \in M$. It is easy to see that if $(X'_i)_{i\in I}$ is a chain of elements of M then $Y' = \bigcup_{i\in I} X'_i$ is an R-set in M. So, we may apply Zorn Lemma and hence M has a maximal element, say Z.

Let N be the set of R-sets X' such that $Y \subseteq X'$ and $X' \cap Z = \emptyset$. By the same argument as above we may apply the Zorn Lemma to N and let Z' be the maximal element of N. So, we have $X \subseteq Z$, $Y \subseteq Z'$ and Z and Z' are R-sets. We shall show that $Z \cup Z' = W$, which will yield that $Z' = W \setminus Z$ and that Z and Z' are good sets.

Let x be an arbitrary element of W. Then by Lemma 6.2, either (i) $Z \cup \equiv(x)$ is an R-set and $(Z \cup \equiv(x)) \cap Z' = \emptyset$ or (ii) $Z' \cup \equiv(x)$ is an R-set and $(Z' \cup \equiv(x)) \cap Z = \emptyset$. In the case of (i), $(Z \cup \equiv(x)) \cap Y = \emptyset$ because $Y \subseteq Z'$. Since $X \subseteq Z \subseteq Z \cup \equiv(x)$ we obtain that $Z \cup \equiv(x) \in M$. By maximality of

Z in M, $x \in Z$. In the case of (ii), $Y \subseteq Z' \subseteq Z' \cup \equiv (x)$, which implies that $Z' \cup \equiv (x) \in N$. By maximality of Z' in N, $x \in Z'$. This completes the proof. **Q.E.D.**

Let $GS(\mathcal{F})$ denote the set of all good sets of the IC-frame $\mathcal{F}$.

LEMMA 6.4. Let $\mathcal{F} = \langle W, \equiv, R \rangle$ be an IC-frame. Then,

(I) $GS(\mathcal{F})$ is non-empty;

(II) $\bigcup_{X \in GS(\mathcal{F})} X = W$.

PROOF: (I) We have seen that $\emptyset$ is an R-set. By applying Theorem 6.3 with $X = Y = \emptyset$, we obtain that there is a good set in $GS(\mathcal{F})$.
(II) Let us show that for any $x \in W$, there is a good set containing x. By Lemma 6.2(IV), $\equiv (x)$ is an R-set. By applying Theorem 6.3 with $X = \equiv (x)$ and $Y = \emptyset$, we get that there is a good set containing x. **Q.E.D.**

LEMMA 6.5. Let $\mathcal{F} = \langle W, \equiv, R \rangle$ be an IC-frame. Then, for any $x, y \in W$,

(I) $x \equiv y$ iff for all $X \in GS(\mathcal{F})$, $x \in X$ iff $y \in X$;

(II) xRy iff for all $X \in GS(\mathcal{F})$, $x \in X$ iff $y \notin X$.

PROOF: (I) ($\rightarrow$) Assume $x \equiv y$ and $X \in GS(\mathcal{F})$. Then obviously, $x \in X$ iff $y \in X$.
($\leftarrow$) Assume for all $X \in GS(\mathcal{F})$, $x \in X$ iff $y \in X$ and suppose not $x \equiv y$. Then, $\equiv (x) \cap \equiv (y) = \emptyset$. By Lemma 6.1(IV), $\equiv (x)$ and $\equiv (y)$ are R-sets. By Theorem 6.3, there is a good set Z such that $\equiv (x) \subseteq Z$ and $\equiv (y) \subseteq W \setminus Z$. So, $x \in Z$ and $y \notin Z$, a contradiction.
(ii) ($\rightarrow$) Suppose xRy and $X \in GS(\mathcal{F})$. Suppose $x \in X$. Then since X is an R-set, we obtain that $y \notin X$. Now, suppose $y \notin X$. So, $y \in W \setminus X$ and $W \setminus X$ is a good set and therefore $x \notin W \setminus X$, that is $x \in X$.
($\leftarrow$) Assume for all $X \in GS(\mathcal{F})$, $x \in X$ iff $y \notin X$ and suppose not xRy. Then by Lemma 6.2(III), $\equiv (x) \cup \equiv (y)$ is an R-set. By applying Theorem 6.3 with $X = \equiv (x) \cup \equiv (y)$ and $Y = \emptyset$, there is a good set Z such that $X \subseteq Z$. So, $x, y \in Z$, which is in contradiction with the assumption. **Q.E.D.**

THEOREM 6.6. (first-order characterization theorem for indiscernibility and complementarity) Each IC-frame is a standard IC-frame over some deterministic information system [resp. over some total information system].

PROOF: Let $\mathcal{F} = \langle W, \equiv, R \rangle$ be an IC-frame. Let $\mathcal{S}$ be the set-theoretical information system $\langle W, V \rangle$ such that $V = \{\{X\} : X \in GS(\mathcal{F})\}$ [resp. $V = \{GS(\mathcal{F})\}$]. By Lemma 6.4(I), V is non-empty. Since each element of V is a singleton, then $\mathcal{S}$ is deterministic [resp. by Lemma 6.4(II), $\mathcal{S}$ is total]. We have to show that $\equiv$ coincides with $\equiv_{\mathcal{S}}$ and R coincides with $R_{\mathcal{S}}$. Applying Lemma 3.1 and Lemma 6.5,

- $x \equiv_{\mathcal{S}} y$ iff for all $X \in V$ and for all $v \in X$, $x \in v$ iff $y \in v$ iff for all $Y \in GS(\mathcal{F})$ $x \in Y$ iff $y \in Y$ iff $x \equiv y$;

- $xR_{\mathcal{S}}y$ iff for all $X \in V$ and for all $v \in X$, $x \in v$ iff $y \notin v$ iff for all $Y \in GS(\mathcal{F})$ $x \in Y$ iff $y \notin Y$ iff xRy.

When $\mathcal{S}$ is total, the end of the proof is similar. **Q.E.D.**

We shall show that indiscernibility and complementarity in property systems (see [Vak90, Vak91]) have the same first-order characterization as in information system. Let us recall the relevant definitions from [Vak91]. A *property system* $\mathcal{S}$ is a triple $\mathcal{S} = \langle OB, PR, f \rangle$ where OB is a non-empty set of objects, PR is a non-empty set of elements called *properties* and f is a function which assigns to each object $x \in OB$ a subset $f(x) \subseteq PR$ called the set of properties of x. The relations of indiscernibility and complementarity derived from a property system $\mathcal{S}$ have the following definitions:

- $x \equiv_{\mathcal{S}} y \stackrel{\text{def}}{\Leftrightarrow} f(x) = f(y)$;
- $xR_{\mathcal{S}}y \stackrel{\text{def}}{\Leftrightarrow} f(x) = PR \setminus f(y)$.

It is easy to see that the frame $\langle OB, \equiv_{\mathcal{S}}, R_{\mathcal{S}} \rangle$ is an IC-frame and the theory of these frames can be applied to prove the following:

Theorem 6.7. Each IC-frame is a standard frame over some P-system.

Proof: Let $\mathcal{F} = \langle W, \equiv, R \rangle$ be an IC-frame. We define a property system $\mathcal{S} = \langle OB, PR, f \rangle$ as follows:

- $OB \stackrel{\text{def}}{=} W$;
- $PR \stackrel{\text{def}}{=} GS(\mathcal{F})$;
- for $x \in W$, $f(x) \stackrel{\text{def}}{=} \{X \in GS(\mathcal{F}) : x \in X\}$.

Then, by using Lemma 6.5 we obtain:

- for all $X \in GS(\mathcal{F})$, $x \in X$ iff $y \in X$ iff for all $X \in GS(\mathcal{F})$, $X \in f(x)$ iff $X \in f(y)$ iff $f(x) = f(y)$ iff $x \equiv_{\mathcal{S}} y$;
- for all $X \in GS(\mathcal{F})$, $x \in X$ iff $y \notin X$ iff for all $X \in GS(\mathcal{F})$, $X \in f(x)$ iff $X \notin f(y)$ iff $f(x) = PR \setminus f(y)$ iff $xR_{\mathcal{S}}y$.

Q.E.D.

7 The information logic LIC for indiscernibility and complementarity

Given a set $\texttt{PRP} = \{\texttt{p}_1, \texttt{p}_2, \ldots\}$ of *atomic formulae*, the formulae $\phi \in \texttt{FML}$ are inductively defined as follows for $\texttt{p}_i \in \texttt{PRP}$:

$$\phi \ ::= \texttt{p}_i \ \mid \ \phi_1 \wedge \phi_2 \ \mid \ \neg\phi \mid \ [\equiv]\phi \ \mid \ [R]\phi$$

Standard abbreviations include $\Leftrightarrow$, $\langle R \rangle$, $\langle \equiv \rangle$.

A *model* $\mathcal{M}$ is a structure $\langle W, \equiv, R, m \rangle$ such that $\mathcal{F} = \langle W, \equiv, R \rangle$ is a member of $\mathcal{KNS}$ and m is a meaning function $m : \texttt{PRP} \rightarrow \mathcal{P}(W)$. $\mathcal{M}$ is said to be based on $\mathcal{F}$. The satisfiability of formulas (written $\mathcal{M}, x \models \phi$) is defined as usual. The logic LIC is a propositional modal logic whose set of formulae is FML and the semantic structures are models based on IC-frames. A formula ϕ is said to be *true* in the model $\mathcal{M} = \langle W, \equiv, R, m \rangle$ (written $\mathcal{M} \models \phi$) $\stackrel{\text{def}}{\Leftrightarrow}$ for all $x \in W$, $\mathcal{M}, x \models \phi$. A formula ϕ is said to be *true* in the frame $\mathcal{F}$ (written $\mathcal{F} \models \phi$) $\stackrel{\text{def}}{\Leftrightarrow}$ ϕ is true in all the models based on $\mathcal{F}$.

Let us define the Hilbert-style system HLIC as follows. The set of axiom schemes consists of the formulas of the following form:

(PC) the tautologies of the Propositional Calculus;

(K) $[\mathsf{a}](\phi \Rightarrow \psi) \Rightarrow ([\mathsf{a}]\phi \Rightarrow [\mathsf{a}]\psi)$ for $\mathsf{a} \in \{R, \equiv\}$;

(A1) $[\equiv]\phi \Rightarrow \phi$; (A2) $\phi \Rightarrow [\equiv]\langle\equiv\rangle\phi$; (A3) $[\equiv]\phi \Rightarrow [\equiv][\equiv]\phi$;

(A4) $\phi \Rightarrow [R]\langle R\rangle\phi$; (A5) $[\equiv]\phi \Rightarrow [R][\equiv]\phi$; (A6) $[\equiv]\phi \Rightarrow [R][R]\phi$.

The inference rules of HLIC are the *modus ponens* (from ϕ and $\phi \Rightarrow \psi$ infer ψ) and *necessitation* (from ϕ infer $[\mathsf{a}]\phi$ for $\mathsf{a} \in \{R, \equiv\}$). We write $\phi \in$ HLIC to denote that ϕ is a theorem of HLIC.

Let $\mathcal{C}$ be a class of non-standard IC-frames. HLIC is *sound* with respect to $\mathcal{C}$ $\stackrel{\text{def}}{\Leftrightarrow}$ for every formula ϕ, if $\phi \in$ HLIC, then ϕ is true in every frame from $\mathcal{C}$. HLIC is *complete* with respect to $\mathcal{C}$ if for every formula ϕ, if ϕ is true in every frame from $\mathcal{C}$ then $\phi \in$ HLIC.

Lemma 7.1. For $\mathcal{C}$ in $\{\mathcal{K}, \mathcal{KS}, \mathcal{KSD}, \mathcal{KST}, \mathcal{KNS}\}$, HLIC is sound with respect to $\mathcal{C}$

8 Completeness of the logic LIC

In this section we investigate completeness of HLIC with respect to the classes of IC-frames defined in section 4.

Theorem 8.1. HLIC is complete with respect to the class $\mathcal{KNS}$ of non-standard IC-frames.

Proof: The proof can be obtained with a standard method of modal logic. The axiom schemes (A1)-(A6) modally define the conditions (S1)-(S6). All the axiom schemes (A1)-(A6) are Sahlqvist's formulas, so completeness with respect to the non-standard semantics is immediate from Sahlqvist's Theorem. **Q.E.D.**

Observe that (A1)-(A6) are equivalent to *primitive modal formulae* in the sense of [Kra96] and therefore a cut-free display calculus exists for the logic LIC and it is sound and complete with respect to the class $\mathcal{KNS}$ of nonstandard IC-frames.

The completeness of LIC with respect to the classes of standard IC-frames can be obtained by applying the copying method introduced in [Vak90].

Definition 8.1. (see e.g. [Vak90]) Let $\mathcal{F} = \langle W, \equiv, R \rangle$ and $\mathcal{F}' = \langle W', \equiv', R' \rangle$ be frames and I be a class of maps from W into W'. I is a *copying* from $\mathcal{F}$ into $\mathcal{F}'$ $\stackrel{\text{def}}{\Leftrightarrow}$

(I1) $W' = \{f(x) : f \in I, x \in W\}$;

(I2) for all $x, y \in W$ and for all $f, g \in I$, if $f(x) = g(y)$, then $x = y$;

(I3) for all $\mathtt{a} \in \{R, \equiv\}$, for all $x, y \in W$ and for any $f \in I$, if $x\mathtt{a}y$, then there is $g \in I$ such that $f(x)\mathtt{a}'g(y)$;

(I4) for all $\mathtt{a} \in \{R, \equiv\}$, for all $x, y \in W$, and for all $f, g \in I$, if $f(x)\mathtt{a}'g(y)$, then $x\mathtt{a}y$.

LEMMA 8.2. [Vak90] Let I be a copying from $\mathcal{F} = \langle W, \equiv, R\rangle$ into $\mathcal{F}' = \langle W', \equiv', R'\rangle$, and $\mathcal{M} = \langle W, \equiv, R, m\rangle$ and $\mathcal{M}' = \langle W', \equiv', R', m'\rangle$ be models such that for $\mathtt{p} \in \mathtt{PRP}$, $m'(\mathtt{p}) = \{f(x) : x \in m(\mathtt{p}), f \in I\}$. Then, for $x \in W$, $f \in I$ and for any formula ψ, $\mathcal{M}, x \models \psi$ iff $\mathcal{M}', f(x) \models \psi$.

The proof is by induction on the complexity of ψ.

LEMMA 8.3. Let $\mathcal{F} = \langle W, \equiv, R\rangle$ be a non-standard IC-frame. Then, there exists an IC-frame $\mathcal{F}' = \langle W', \equiv', R'\rangle$ and a copying from $\mathcal{F}$ to $\mathcal{F}'$.

PROOF: Let $\mathcal{F}' = \langle W', \equiv', R'\rangle$ be the frame defined as follows:

- $W' \stackrel{\text{def}}{=} W \times \{-1, 1\}$;
- for any $\langle x, i\rangle, \langle y, j\rangle \in W'$, $\langle x, i\rangle \equiv' \langle y, j\rangle \stackrel{\text{def}}{\Leftrightarrow} x \equiv y$ and $i = j$;
- for any $\langle x, i\rangle, \langle y, j\rangle \in W'$, $\langle x, i\rangle R' \langle y, j\rangle \stackrel{\text{def}}{\Leftrightarrow} xRy$ and $i = -j$.

Let $I = \{f_{-1}, f_1\}$ be the set of maps such that for $i \in \{-1, 1\}$, $f_i : W \to W'$ and for $x \in W$, $f_i(x) = \langle x, i\rangle$. It is easy to see that $\mathcal{F}'$ is an IC-frame (satisfying the conditions (S1)-(S7)) and that I is indeed a copying. The satisfaction of (I1), (I2) and (I4) is straighforward. By way of example, let us show that (I3) holds with $\mathtt{a} = R$. Let xRy and $i \in \{-1, 1\}$. So, $f_i(x)R'f_{-i}(y)$. **Q.E.D.**

THEOREM 8.4. HLIC is complete with respect to the class $\mathcal{K}$ of IC-frames.

PROOF: Suppose ϕ is not a theorem of HLIC. Then by Theorem 8.1, ϕ is not true in some non-standard IC-frame $\mathcal{F} = \langle W, \equiv, R\rangle$, i.e. there is a model $\mathcal{M} = \langle W, \equiv, R, m\rangle$ based on $\mathcal{F}$ and $x \in W$ such that $\mathcal{M}, x \not\models \phi$. By Lemma 8.3, there is an IC-frame $\mathcal{F}' = \langle W', \equiv', R'\rangle$ and a copying I from $\mathcal{F}$ to $\mathcal{F}'$. Then by Lemma 8.2, $\mathcal{M}', x \not\models \phi$ where $\mathcal{M}'$ is a model based on $\mathcal{F}'$ such that for any propositional variable $\mathtt{p}$, $m'(\mathtt{p}) = \{f(x) : x \in m(\mathtt{p}), f \in I\}$. So, ϕ is not true in the class of IC-frames. **Q.E.D.**

Theorem 8.5 below is one of the main results in this paper:

THEOREM 8.5. (Standard completeness) Let ϕ be a formula. The statements below are equivalent:

(I) ϕ is a theorem of HLIC;

(II) ϕ is true in every frame from $\mathcal{KS}$;

(III) ϕ is true in every frame from $\mathcal{KSD}$;

(IV) ϕ is true in every frame from $\mathcal{KST}$;

PROOF: (I) → (II), (I) → (III) and (I) → (IV) are true by Lemma 7.1.
(II) → (III) Every IC-frame derived from a deterministic information system is a standard frame.
(II) → (IV) Every IC-frame derived from a total information system is a standard frame.
For the implications (III) → (I) [resp. (IV) → (I)], suppose ϕ is not a theorem of HLIC. By Theorem 8.4, ϕ is not valid in some IC-frame $\mathcal{F} = \langle W, \equiv, R\rangle$. By Theorem 6.6, $\mathcal{F}$ is a standard frame over some deterministic [resp. total] information system. This proves (III) → (I) [resp. (IV) → (I)], which completes the proof. **Q.E.D.**

9 Complexity of the satisfiability problem for LIC

Lemma 9.1 is used in the proof of Lemma 9.2.

LEMMA 9.1. Let $\mathcal{F} = \langle W, \equiv, R\rangle$ be an IC-frame, $x \in W$ and $\mathcal{F}_x = \langle W_x, \equiv_x, R_x\rangle$ be the generated subframe from x. Then

(I) If there is an $y \in W$ such that xRy, then $W_x = \equiv(x) \cup \equiv(y)$ and for all $x', y' \in W_x$,
- $x' \equiv_x y'$ iff either $\{x', y'\} \subseteq \equiv(x)$ or $\{x', y'\} \subseteq \equiv(y)$;
- $x'R'y'$ iff either $x' \in \equiv(x)$ and $y' \in \equiv(y)$ or $x' \in \equiv(y)$ and $y' \in \equiv(x)$;

(II) if there is no $y \in W$ such that xRy, then $W_x = \equiv(x)$, $\equiv_x = W_x \times W_x$ and $R_x = \emptyset$.

LEMMA 9.2. Any LIC-satisfiable formula ϕ is satisfiable in a LIC-model $\langle W, \equiv, R, m\rangle$ based on an IC-frame $\langle W, \equiv, R\rangle$ such that $card(W) \leq 2 \times |\phi| + 2$.

Here $|\phi|$ denotes the length of the formula ϕ, that is the number of symbols occurring in ϕ.
PROOF: Let $\mathcal{M} = \langle W, \equiv, R, m\rangle$ be an LIC-model, $w_0 \in W$ such that $\mathcal{M}, w_0 \models \phi$.
Case 1: $R(w_0) \neq \emptyset$
So let w_1 be some element in $R(w_0)$. Let X_0^ϕ be the set

$$X_0^\phi \stackrel{\text{def}}{=} \{\psi : [\equiv]\psi \in sub(\phi),\ \mathcal{M}, w_0 \not\models [\equiv]\psi\} \cup \{\psi : [R]\psi \in sub(\phi),\ \mathcal{M}, w_1 \not\models [R]\psi\}$$

where $sub(\phi)$ denotes the set of subformulae of ϕ. Let X_1^ϕ be the set

$$X_1^\phi \stackrel{\text{def}}{=} \{\psi : [R]\psi \in sub(\phi),\ \mathcal{M}, w_0 \not\models [R]\psi\} \cup \{\psi : [\equiv]\psi \in sub(\phi),\ \mathcal{M}, w_1 \not\models [\equiv]\psi\}$$

For each $\psi \in X_i^\phi$ ($i \in \{0, 1\}$), we choose some witness $w_i^\psi \in \equiv(w_i)$, such that $\mathcal{M}, w_i^\psi \not\models \psi$. Let $\mathcal{M}' = \langle W', \equiv', R', m'\rangle$ be the restriction of $\mathcal{M}$ to

$$W' = \{w_0, w_1\} \cup \{w_0^\psi : \psi \in X_0^\phi\} \cup \{w_1^\psi : \psi \in X_1^\phi\}$$

Since the conditions (S1)-(S7) are universally quantified first-order formula, by Łos-Tarski preservation theorem, the class of IC-frames is closed under subframes. So $\mathcal{M}'$ is a model based on an IC-frame and for any $i \in \{0,1\}$ and for any $w' \in \equiv' (w_i)$, $w'' \in \equiv' (w_{1-i})$, $w' R' w''$. Let us show by induction on the structure of formulae that for any $\psi \in sub(\phi)$ and for any $w' \in W'$, $\mathcal{M}, w' \models \psi$ iff $\mathcal{M}', w' \models \psi$. We omit the cases when ψ is an atomic proposition or when the outmost connective of ψ is Boolean.

Case 1.1: $\psi = [\equiv]\psi'$

Assume $\mathcal{M}, w' \models \psi$ for some $w' \in W'$. So for all $w'' \in \equiv (w')$, $\mathcal{M}, w'' \models \psi'$. *A fortiori*, for all $w'' \in \equiv' (w')$, $\mathcal{M}, w'' \models \psi'$. By induction hypothesis, for all $w'' \in \equiv' (w')$, $\mathcal{M}', w'' \models \psi'$. So, $\mathcal{M}', w' \models \psi$. Now assume, $\mathcal{M}, w' \not\models \psi$ for some $w' \in W'$. Let $i \in \{0,1\}$ be such that $w' \in \equiv (w_i)$. So there is $w'' \in \equiv (w_i)$ such that $\mathcal{M}, w'' \not\models \psi'$. By construction, $\mathcal{M}, w_i^{\psi'} \not\models \psi'$. Since, $w_i^{\psi'} \in W'$, $\mathcal{M}', w_i^{\psi'} \not\models \psi'$ by induction hypothesis. By observing that $w' \equiv' w_i^{\psi'}$, we have $\mathcal{M}', w' \not\models \psi$.

Case 1.2: $\psi = [R]\psi'$

Assume $\mathcal{M}, w' \models \psi$ for some $w' \in W'$. So for all $w'' \in R(w')$, $\mathcal{M}, w'' \models \psi'$. *A fortiori*, for all $w'' \in R'(w')$, $\mathcal{M}, w'' \models \psi'$. By induction hypothesis, for all $w'' \in R'(w')$, $\mathcal{M}', w'' \models \psi'$. So, $\mathcal{M}', w' \models \psi$. Now assume, $\mathcal{M}, w' \not\models \psi$ for some $w' \in W'$. Let $i \in \{0,1\}$ be such that $w' \in \equiv (w_i)$. So there is $w'' \in \equiv (w_{1-i})$ such that $\mathcal{M}, w'' \not\models \psi'$. By construction, $\mathcal{M}, w_{1-i}^{\psi'} \not\models \psi'$. Since, $w_{1-i}^{\psi'} \in W'$, $\mathcal{M}', w_{1-i}^{\psi'} \not\models \psi'$ by induction hypothesis. By observing that $w' R' w_{1-i}^{\psi'}$, then $\mathcal{M}', w' \not\models \psi$.

Case 2: $R(w_0) = \emptyset$. Let X^ϕ be the set $X^\phi = \{\psi : [\equiv]\psi \in sub(\phi),\ \mathcal{M}, w_0 \not\models [\equiv]\psi\}$. For each $\psi \in X^\phi$, there is some *witness* $w^\psi \in \equiv (w_0)$ such that $\mathcal{M}, w^\psi \not\models \psi$. One can show that $R(w^\psi) = \emptyset$. w^ψ is a witness of that fact that $\mathcal{M}, w \not\models [\equiv]\psi$. Let $\mathcal{M}' = \langle W', \equiv', R', m' \rangle$ be the restriction of $\mathcal{M}$ to $W' = \{w_0\} \cup \{w^\psi : \psi \in X^\phi\}$. That is $R' \stackrel{\text{def}}{=} R \cap W' \times W'$, $\equiv' \stackrel{\text{def}}{=} \equiv \cap W' \times W'$ and for any atomic proposition $\mathtt{p}$, $m'(\mathtt{p}) \stackrel{\text{def}}{=} m(\mathtt{p}) \cap W'$. Observe that $R' = \emptyset$. One can show by induction on the structure of formulae that for any $\psi \in sub(\phi)$ and for any $w' \in W'$, $\mathcal{M}, w' \models \psi$ iff $\mathcal{M}', w' \models \psi$. Consequently, $\mathcal{M}', w_0 \models \phi$ and observe that $\mathcal{M}'$ is also a model based on an IC-frame. The case 2 is indeed similar to the proof of [Lad77, Lemma 6.1].

Q.E.D.

In the proof of Lemma 9.2, we select a submodel that contains enough witnesses for the satisfaction of the diamond formulae. This technique has been already successfully applied to other logics such as in [Lad77, Dem98]. Given a finite structure $\langle W, \equiv, R, m \rangle$ and a formula ϕ, it is known that checking whether ϕ is satisfiable in $\langle W, \equiv, R, m \rangle$ can be done in time $\mathcal{O}(card(W)^2 \times |\phi|)$. Consequently,

THEOREM 9.3. The set of formulas that are true in all the IC-frames is decidable.

Observe that there exists a linear-time transformation from LIC-validity into F0^3-validity (the fragment of classical logic using only three individual variables) (see e.g. [Ben83]). However the exact fragment delineated by the relational translation belongs neither to $F0^2$ [Mor75] nor to the loosely guarded fragment that are known to be decidable fragments [Mor75, ANB98].

THEOREM 9.4. LIC-satisfiability is in **NP**, that is it can solved in polynomial-time by a non-deterministic Turing machine.

Observe that to check that a structure $\langle W, \equiv, R\rangle$ is an IC-frame can be done in **P**-time. **NP**-hardness of LIC-satisfiability is simply due to the fact that the LIC-satisfiability problem contains SAT (satisfiability problem for the classical propositional calculus). The complexity upper bound from Theorem 9.4 should be compared with complexity lower bounds of other known bimodal logics (see e.g. [HM92, Spa93]) that can be for instance **PSPACE**-hard (for the bimodal logic with exactly two independent S5 modal operators) or **EXPTIME**-hard (for K plus the universal modal operator).

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