The Game of Chaos

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Abstract

Game of Chaos is a red sorcery in the oldest of all customizable card games: **Magic, the Gathering**, trademarked by the Wizards of the Coast inc. Successfully casting this spell enables the caster to engage the opponent player in a potentially unbounded series of coin-flip games about life. Initially the ante is one life. The winner decides to stop or to play a next round. However, for every next round the ante in lives is doubled. This will ensure that the game will be terminated as soon as the loser has his total amount of lives reduced to zero or lower, since terminating the game of chaos at this point yields immediate victory of the duel.

Given the inherent symmetry of this game the question is whether it offers the caster any strategic advantage to play it. For every possible play which yields a positive outcome there is a corresponding play which yields the same outcome to his opponent. Consequently the utility value of this game should be zero.

We invoke elementary game theory in order to illustrate how this theory does confirm this intuition. However, the same theory can also be invoked in different scenarios, like *Thorgrim's last stand* where the utility value can be shown to be positive.

The illustrations for this note are contained in the powerpoint presentation which is available in pdf format at the website of the first author. See The game of Chaos for 12 pages displaying the sheets presented at the 1999 Dutch Mathematical Conference in Utrecht.

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1 Game of Chaos - why play it at all?

Game of Chaos is a red sorcery in the oldest of all customizable card games: **Magic, the Gathering**, trademarked by the Wizards of the Coast inc. The card is shown on the sheet The game of Chaos. For readers not familiar with this game: it is a customizable card game which means that every player composes a deck of cards to play with from a large collection of available cards. Purpose of the game is to defeat your opponent by reducing his number of lives (initially 20) to zero or less. You can damage your opponent by summoning creatures which may attack your opponent or by casting other spells which will harm him or her in some other way. You defend yourself against your opponent by summoning creatures which will block the attacking creatures (and which may get killed, destroyed or buried in the process), by countering spells cast by your opponent, by casting spells which will protect you (or your creatures), or by invoking effects which will repair and heal damage suffered previously in a turn, or which may damage your opponent or his or her creatures. Each spell you want to cast must correspond to a card you currently have in your hand, and it can be cast only once. The cost of using a card (casting cost) is expressed in units of mana, which is obtained by "tapping" lands you have played earlier in the game. Mana exists in five colored flavors (red, white, black, green and blue) and a generic version. The same color labels are ascribed to most spells; a red spell will require an amount of red mana together with another amount of generic mana. Hence it is not only a matter of having enough lands or other mana sources; they also should be of the right type.

Overall the rules of the game are quite complex, and subject to regular revisions, partly due to the introduction of series of expansion cards, and partly due to an increased sensitivity for potential weak spots in the game. At present the sixth edition of the game is about to appear. However the basic principle of the game is expressed by the slogan that for almost every rule in the game there exists a card generating an exception against this rule when successfully cast. A rather complete set of rules for the fourth edition can be found in [2].

The *Game of Chaos* is just a single card in the game, which introduces in fact a subgame which may, but doesn't necessarily terminate the entire game. Successfully casting this spell enables the caster to engage the opponent player in a potentially infinite series of coin-flip games about life. Initially the ante is one life. The player flips a coin and the opponent calls head or tails while the coin is in the air. If the outcome is correct the player gains one life and the opponent loses a life. Subsequently the winner decides to stop or to play a next round. However, for every next round the ante in lives is doubled. This will ensure that the game will be terminated as soon as the loser has his total amount of lives reduced to zero or lower, since terminating the game of chaos at this point yields immediate victory of the duel (disregarding for the moment the possible effect of damage prevention or healing spells in the game. In fact under the new rules for the sixth edition of the game this problem evaporates: one no longer has to wait till the end of a phase in order to decide whether you are dead by having zero lives or less).

In the sequel the two players will be identified by their names **Thorgrim** and **Urgat**. Thorgrim is High King of the Dwarfs, whereas Urgat is an Orc Big Boss, both originating from the Warhammer world[1]; they have the perfect characteristics of opponents in a game: they have been involved in a feud which has lasted for over a millennium, and they hate each other. Our two friends are illustrated on sheet the players.

Developing the game tree (incorporating the alternation between chance moves and deterministic moves by the two players), yields a highly symmetric structure. At a deterministic node the player who has to move can decide to terminate the game with payoff $+n/-n$, where n is the cumulative number of lives gained by the Thorgrim. If the player decides however to continue the next node is a chance move which has two descendant nodes (each with local probability 1/2) which are deterministic, where the player who wins the coin flip is to move. See the illustration on sheet the game tree, where we actually have reduced the size of the tree by an abbreviation hiding the chance nodes.

It is easy to see that the payoff is always odd and positive for the player who has won the last coin flip. But for every play with positive payoff $+n/-n$ for Thorgrim, there exists a reflected play resulting in payoff $-n/+n$ in favor of Urgat. Moreover both plays occur with equal probability. On behalf of this symmetry it seems that the expected value of the game is zero: there is no rational reason to play it. Certainly it is highly irrational to pay the casting cost of three red mana, if you know that the same amount of mana could inflict 9 damage to your opponent by casting three lightning bolts. Alternatively one could summon three 1/1 goblins[2], or a 1/1 goblin together with a goblin king: a 2/2 creature with the special ability *all goblins get +1/+1 and mountainwalk*, which means that these goblins become unblockable when attacking a player controlling mountains, the red type of land. So why would a player even consider to insert this card in his deck?

Regular players of Magic, the Gathering will indicate two possible scenario's where the Game of Chaos seems to make sense: you can play this card if one has more lives than your opponent; alternatively, you can play this card as a *last stand* if you are faced with certain defeat in the next turn of the opponent (you are defenseless against a superior force of flying creatures with trample, and the Game of Chaos is the last card in your hand). These judgements are based on naive intuition only. The question is whether one can perform some sort of computation validating these beliefs.

In this note I want to provide an affirmative answer to this last question. A simple application of the von Neumann - Morgenstern *Utility Theory* illustrates that the expected value of the game is zero in a symmetric starting position like at the start of the game. On the other hand it will ascribe a positive value to the (sub)game in the two scenario's suggested.

[1]Warhammer is a trademark of the Games Workshop

[2]in the notation a/b the number a denotes the attack strength which represents the amount of damage the creature does if it attacks, and b denotes the toughness representing the amount of damage the creature has to absorb in order to be destroyed

2 Von Neumann - Morgenstern Utility Theory

For the purpose of this note a game is a finite rooted tree. Internal nodes are labeled either by the players $\{T, U\}$, indicating the player who has to move at this position, or the node is labeled to be a chance move (label C) in which case the outgoing edges have probabilities assigned which should sum up to 1. In our sheets these labels are indicated by colors: Thorgrim's node are red; Urgat's node are dark green and the chance nodes are light green. The leaves of the tree are labeled by pay-offs for both players. For the purpose of this note we focus on *strictly competitive zero sum games* where each pay-off has the form $x/-x$ for some real value x.

In the procedure of Backward Induction pay-off values are assigned to intermediate nodes as well. At a node labeled T or U the player who has to move will choose the descendent node with the highest pay-off to that player, and this will result into the well known min-max algorithm. At a chance node the reasonable pay-off is the expected pay-off at the descendant nodes. In the case of a zero-sum game the two operations preserve the zero-sum format, and hence the procedure is well defined, assigning eventually a value to the root node which then becomes the value of the game.

The problem is that in general the pay-off values are introduced to represent preferences rather than absolute values. If outcome X is preferred by Thorgrim over outcome Y one can ascribe to X a higher *utility* value x than the value y assigned to Y. Given a set of possible outcomes ordered by Thorgrim's preferences there exist many order preserving utility assignments all representing the same preferences. However these utility assignments will ascribe different preferences to the expected utilities computed at chance nodes in the game tree.

One can also look at this situation from the perspective of the strategic form of the game. A *pure strategy* of a player selects for every node where he or she has to play a move selecting one of the descendants in the tree. If one applies a pair of strategies for both players to a game, truncating moves which are never made under these strategies a game with only chance moves remains. Such games are in fact Compound Lotteries where the individual outcomes occur with probabilities summing up to 1. For the application of backward induction the problem is now how to compare two of these lotteries. That this is a severe problem is illustrated by the notorious example constructed by Allais; see sheet Comparing Complex Lotteries; Allais Example. The crux of this example is that it is "irrational" to prefer the left lottery over the right one in the first row and have converse preferences in the second row; which is told to be a frequently observed behavior among the natives.

Von Neumann - Morgenstern Utility Theory provides us with a strategy to overcome these problems. The key observation is that the problem doesn't arise in case there exist only two possible outcomes for the game: winning or loosing. These outcomes can be scaled to the values 1 and 0. Next one can ascribe to an intermediate outcome X such that Thorgrim prefers winning over X and X over loosing, a *utility* value q such that Thorgrim is indifferent between X and participating in a *lottery* with probability q of winning and $1-q$ of loosing. The utility value q reflects Thorgrim's taste and appreciation for the outcome X. See sheet Utility Intermediate Outcome. Systematic substituting intermediate outcomes by these lotteries in a game truncated after a choice of strategy for both players yields a so-called *compound lottery* which can be simplified to a simple one. And since these simple lotteries have two outcomes only, they can easily be compared: the preferred lottery is the one with the greater chance of winning, I.E., the one with the higher utility. Moreover, the computation rule of taking the expected utility at a chance node is consistent with this interpretation of intermediate outcomes. See sheet Utility Lottery = Expected Utility Outcomes.

So what the von Neumann - Morgenstern Theory requires us to do is first to assign to intermediate outcomes a utility value which reflects the taste of the player. Subsequently values of the nodes in the tree can be obtained by the calculation rule of backward induction, using expected utility at chance nodes.

It can be shown that the utility values reflecting the taste of a player are unique up to an affine scaling. One can normalize utilities by assigning the best and the worst outcome utilities 0 and 1 respectively.

We have introduced this theory in the context of a strictly competitive zero sum game, but these restrictions turn out to be unnecessary. One just has to assign separate utility values for both players to each outcome, and at chance nodes expected utilities are assigned to both players. At a deterministic node the player who is to move selects the node with the highest utility for that player.

Note that even in a situation where X in fact represents an amount of money or other quantifiable goods, there is no a priori reason why the relation between this quantity and the appreciation as expressed by the utility should be represented by a affine or linear function. The only rationality condition which one should enforce is that less money should not be preferred over more: the relation should be at least monotone non-decreasing.

3 Utilities for the game of Chaos

In the game tree of the game of Chaos we can take each position where one of the players has lost all his available lives to be lost for that player and won for the other. The remaining positions where both players are still alive yield intermediate outcomes in case the player who has won the last coin-flip decides to terminate. To all positions we should assign utilities in order to apply the von Neumann - Morgenstern Utility theory. By selecting appropriate assignments we can describe scenarios where playing the game is meaningless but also alternatives where the game obtains a positive value for a player.

In the backward induction computation a player will compare at a node where he is to move the utility of the intermediate outcome collected by terminating with the average of the two utilities corresponding to winning or loosing the next coin flip. As long as both players use Linear Utilities which are linearly dependent on the number of lives for the players, the outcome of this comparison will be a comparison between equal utilities due to the fact that continuing in a next coin flip yields a fifty/fifty chance of collecting the ante for the next round. It follows that in all scenario's where the utility for a player of an outcome is linear in his or her number of lives at this outcome the value of the game vanishes: each player will assign the same utility to playing a next round and terminating. An example of such a utility function is the function u_1 which assigns in a position where Thorgrim (Urgat) has gained m lives the utility m to Thorgrim (Urgat). So no player will assign positive value to playing the game and therefore he will not spend the required mana to cast this sorcery.

The more interesting scenario's are obtained by disturbing the linear relation between utility and number of lives. An evident attempt is to ascribe utility 1 to a position where the other player is dead, -1 to a position where the player has lost all his lives, and 0 to all intermediate positions. Go for the kill! This function is called u_2. In sheet Go for the kill! the game tree has been evaluated for a modified game where both players initially have 5 lives. Some of the intermediate nodes obtain non-zero utility but the overall value of the game remains zero.

The symmetry can not be broken by assigning different utilities to both players. In sheet Mixed Utilities I illustrate the game tree where Thorgrim uses number of lives gained for utility and Urgat is going for the kill. Once again the net value of the game is zero for both players.

The next attempt is to make both players insensitive towards losing the game. Utility function u_3 assigns value 1 to a position where a player has won and value 0 to all other positions. In sheet Winning is all I present the game tree starting from the initial situation where both players have 5 lives. The resulting utility is 0.5 for both players representing their 50% chance of winning the game of Chaos.

If the initial situation is asymmetrical the value of the game becomes positive or negative depending on whether the player has more or less lives than his opponent. In sheet Unequal Start I illustrate the game tree for an initial situation where Thorgrim has 6 lives and Urgat 4. The game has value $1/8$ for Thorgrim and $-1/8$ for Urgat. The utility function used is u_2.

Our final scenario is the situation where Thorgrim has one life remaining against 6 for Urgat and must win the game of Chaos in order to survive. Thorgrim uses utility u_3 and Urgat u_2. The outcome is that the value of the game is $1/8$ for Thorgrim representing the fact that he must win three coin-flips in a row in order to survive, and $3/4$ for Urgat illustrating the excess of his chance of winning ($7/8$) over the chance of $1/8$ of losing.
See sheet Thorgrim's last stand.

I hope that these simple illustrations suffice to convince Johan of the usefulness of the von Neumann - Morgenstern Utility Theory. Remains to apply it to games in logic.

References

[1] Ken Binmore, *Fun and Games; a text on Game Theory*, D.C Heath and Company, Lexington, Mass (1992)

[2] Redman. R. et. al., *Magic the Gathering; the pocket players guide for the fourth edition.*, Wizards of the Coast, Inc. USA, 1995.