The Modal Completeness of ILW

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Abstract

This paper contains a completeness proof for the system **ILW**, a rather bewildering axiom system belonging to the family of interpretability logics. We have treasured this little proof for a considerable time, keeping it just for ourselves. Johan's fiftieth birthday appears to be the right occasion to get it out of our wine cellar.

Contents

1 Introduction

In interpretability logic the logical properties of the notion of interpretability are studied in much the same way as the logical properties of the notion of provability are studied in provability logic. In the latter a one-place operator $\Box$ is added to the language of propositional logic. The intended meaning in the context of an arithmetical theory T of a formula $\Box A$ is 'A is provable in T' (where A represents an arithmethic formula). In the former, a binary operator $\rhd$ is added to the language of propositional logic. Here the intended meaning of $A \rhd B$ (read 'A interprets B') in an arithmetical theory T is: '$T+B$ is interpretable in $T+A$'.

Interpretability logic extends provability logic: $\Box$ is definable in terms of $\rhd$ via the equation $\Box A =_{def} \neg A \rhd \bot$. Thus, in principle, interpretability logic can disclose at least as much about the underlying arithmetical theory T as provability logic can. Actually, it does disclose more. Provability is a stable notion, interpretability is not. All extensions of $\mathrm{I}\Delta_0 + \mathrm{EXP}$ have the same provability logic. But as it turns out, the interpretability logic **ILM** of Peano Arithmetic differs widely from the interpretability logic **ILP** of ACA_0 (the arithmetical counterpart of Gödel-Bernays set theory).

All interpretability logics studied so far are extensions of the core system **IL**, which is given by the derivation rules Modus Ponens and Necessitation and the axioms $\Box A \to \Box\Box A$ and $\Box(\Box A \to A) \to \Box A$ (Löb's Axiom) of the provability system **L**, plus the axioms:

(J1) $\Box(A \to B) \to (A \rhd B)$
(J2) $(A \rhd B) \wedge (B \rhd C) \to (A \rhd C)$
(J3) $(A \rhd C) \wedge (B \rhd C) \to (A \vee B \rhd C)$
(J4) $(A \rhd B) \to (\Diamond A \to \Diamond B)$
(J5) $\Diamond A \rhd A$

(With respect to priority of parentheses $\rhd$ is treated as $\to$.)

By adding the scheme $(A \rhd B) \to (A \wedge \Box C \rhd B \wedge \Box C)$ one gets the system **ILM** mentioned above. **ILP** is given by **IL** plus the scheme $(A \rhd B) \to \Box(A \rhd B)$. Central in this paper is a third extension of **IL**, the system **ILW** described e.g. in Visser [7]. **ILW** = **IL** + W, where W is the axiom scheme

$$(A \rhd B) \to (A \rhd B \wedge \Box\neg A)$$

The system **ILW** is contained in both **ILM** and **ILP** (see de Jongh-Veltman [2], or Visser [6]), and was at some point conjectured to embody the principles common to all "reasonable" arithmetics. In the meantime, however, Albert Visser discovered two new general principles, M_0: $A \rhd B \to (\Diamond A \wedge \Box C \rhd B \wedge \Box C)$ (see Visser [7]), and P_0: $A \rhd \Diamond B \to \Box(A \rhd B)$ (see Joosten [5]).

¿From a purely modal point of view, it seems wise to first take a proper look at **ILW**, before trying to get to grips with a system like $\mathbf{ILWM_0P_0}$. Indeed, as the completeness proof presented below will show, **ILW** already poses so many problems that the predicate 'bewildering' comes to mind.

2 Semantics

It is a well-known fact that the modal logic **L** is complete with respect to the **L**-frames $\langle W, R\rangle$, which consist of a set of worlds W together with a transitive conversely well-founded relation R.

Definition 1 *If $\langle W, R\rangle$ is a partially ordered set and $w \in W$, then*
$wR = \{w' \in W \mid wR\,w'\}$.

Definition 2 *An* **IL**-*frame is a* **L**-*frame $\langle W, R\rangle$ with an additional relation S_w for each $w \in W$, which has the following properties:*
(i) *S_w is a relation on $w\,R$,*
(ii) *S_w is reflexive and transitive,*
(iii) *if $w', w'' \in w\,R$ and $w'\,R\,w''$, then $w' S_w w''$.*

We will often write S for $\{S_w \mid w \in W\}$.

Definition 3 *An* **IL**-*model is given by an* **IL**-*frame $\langle W, R, S\rangle$ combined with a forcing relation with the clauses:*
(i) $u \Vdash \Box A$ *iff* $\forall v(uRv \Rightarrow v \Vdash A)$,
(ii) $u \Vdash A \rhd B$ *iff* $\forall v(u\,R\,v$ *and* $v \Vdash A \Rightarrow \exists w(vS_u w$ *and* $w \Vdash B))$.

Definition 4
(a) *For $F = \langle W, R, S\rangle$, we write $F \Vdash A$ iff $w \Vdash A$ for every $\Vdash$ on F and every $w \in W$.*
(b) *If* **K** *is a class of frames, we write* $\mathbf{K} \Vdash A$ *iff* $F \Vdash A$ *for each* $F \in \mathbf{K}$.
(c) $\mathbf{K}_W$ *is the class of* **IL**-*frames with the additional property*
(iv) *for any w, the converse of $R \circ S_w$ is well-founded.*

The next lemma states that the scheme W *characterizes* the class of frames $\mathbf{K}_W$.

Lemma 5
(a) *For each A, if $\vdash_{\mathbf{IL}} A$, then $F \Vdash A$.*
(b) $F \Vdash \mathbf{ILW}$ *iff* $F \in \mathbf{K}_W$ (**ILW** *characterizes* $\mathbf{K}_W$).

Proof. Straightforward. $\dashv$

3 Modal completeness

The usual method in modal logic for obtaining completeness proofs is to construct directly or indirectly the necessary countermodels by taking maximal consistent sets of the logic under consideration as the worlds of the model. There are three problems with this approach here. First, there is a problem deriving from the modal logic **L** which is the basis of our system. This logic is not compact: some infinite syntactically consistent sets of formulae are semantically incoherent. A solution is to restrict the maximal consistent sets to subsets of some finite set of formulae. Such a so-called adequate set has to be rich enough to prove the analogon of the valuation lemma which states that a formula A belonging to the adequate set is forced in a world w iff $A \in w$. Therefore it has to be closed under forming of subformulae and single negations. Furthermore, for each particular logic, additional requirements on the adequate set will be needed to be able to apply the axioms.

It turns out that for **ILW** we need the following.

Definition 6 *An* adequate *set of formulae is a set Φ which fulfills the following conditions:*

(i) *Φ is closed under the taking of subformulae,*

(ii) *if $B \in \Phi$, and B is not a negation, then $\neg B \in \Phi$,*

(iii) *$\bot \rhd \bot \in \Phi$,*

(iv) *if B as well as C are the antecedent or consequent of some $\rhd$-formula in Φ, then $B \rhd C \in \Phi$.*

It is not difficult to see that each finite set Γ of formulae is contained in a finite adequate set Φ.

In this connection, we consider $\Diamond A$ to be shorthand for $\neg(A \rhd \bot)$, and $\Box A$ short for $\neg A \rhd \bot$ (unless A is $\neg B$; then $\Box A$ stands for $B \rhd \bot$), so that we can ignore $\Box$- and $\Diamond$-formulas in inductions. Note that in this way we can be sure that, if $B \rhd C$ is a member of an adequate set, then so are $\Diamond B$ and $\Diamond C$.

As usual, we define $\Gamma \prec \Delta \Leftrightarrow$ (i) for each $\Box A \in \Gamma$, it holds that $\Box A, A \in \Delta$, and (ii) for some $\Box A \notin \Gamma$, it holds that $\Box A \in \Delta$. Whenever $\Gamma \prec \Delta$, we say that Δ is a *successor* of Γ. The following lemma transfers from **L** to **IL** and its extensions.

Lemma 7 *Let Γ_0 be a maximal* **ILW***-consistent subset of some finite adequate Φ, and let W_{Γ_0} be the smallest set such that* (i) $\Gamma_0 \in W$ (ii) *if $\Delta \in W$ and Δ' is a maximal* **ILW***-consistent subset of Φ such that $\Delta \prec \Delta'$, then $\Delta' \in W$. Then*

(i) *$\prec$ is transitive and irreflexive on W_{Γ_0},*

(ii) *for each $\Gamma \in W_{\Gamma_0}$, $\Box A \in \Gamma \Leftrightarrow A \in \Delta$ for every Δ such that $\Gamma \prec \Delta$.*

The model supplied by this lemma works fine in a completeness proof for **L**, but it is much too small for **IL** and its extensions. It is not always possible endow $\langle W_{\Gamma_0}, \prec \rangle$ with relations S_Γ for every $\Gamma \in W_{\Gamma_0}$ in such a way that (i) S has all the properties required, and (ii) the valuation lemma can be proved for $\rhd$-formulas. We can no longer identify a world with the set of formulas true in it. In the eventual model it will often occur that different worlds are described by the same maximal consistent **ILW**-consistent subset of Φ. Serious duplication of worlds is necessary already in the case of **IL**. (See de Jongh-Veltman [2] or Japaridze-de Jongh [3] for more explanation on this point.)

To overcome this second problem we need some more machinery.

Definition 8 *Let Γ and Δ be maximal consistent subsets of Φ and let $C \in \Phi$.*
(a) Δ is a C-critical successor of Γ iff

(i) *$\Gamma \prec \Delta$,*

(ii) $\neg B, \Box\neg B \in \Delta$ *for each* B *such that* $B \rhd C \in \Gamma$.

(b) C admits B *with respect to* Γ *iff* B *occurs in some* C*-critical successor of* Γ.

It is easily seen that successors of C-critical successors of Γ are C-critical successors of Γ.

In the model that we are going to build every world w is associated with a maximal **ILW**-consistent subset Γ of some adequate Φ. This set Γ is supposed to give a partial description of w. To ensure that a formula of the form $\neg(B \rhd C) \in \Gamma$ is indeed true in w, the model has to provide a world w' associated with a C-critical successor of Γ containing B. Moreover, all the worlds accessible from w' by the relation S_w should be associated with C-critical successors, too.

Is this feasible? The next two lemmata say it is.

Lemma 9 *Let* Γ *be maximal* **ILW***-consistent in* Φ*, and suppose* $(B \rhd C) \in \Phi$*. Then* $\neg(B \rhd C) \in \Gamma$ *iff* C *admits* B *with respect to* Γ*.*

Proof. From right to left: this follows almost immediately from the definition. ¿From left to right: the proof of lemma 3.6 in de Jongh-Veltman [2] (or lemma 13.12 of Japaridze-de Jongh [3]) applies to **ILW**. $\dashv$

Lemma 10 *Let* Γ *be maximal* **ILW***-consistent in* Φ *and suppose* $(A \rhd D) \in \Gamma$*. If* C *admits* A *with respect to* Γ*, then* C *admits* D*, too.*

Proof. Almost directly from the previous lemma. $\dashv$

The lemmata just mentioned enable us to construct an **IL**-countermodel to A for every A such that $\nvdash_{\mathbf{ILW}} A$. They allow us to connect a so-called *C-critical cone* above w with every world w introduced in the model and with every C such that some $\neg(B \rhd C)$ should be true in w. The worlds in this C-critical cone are all associated with a C-critical successor of the set of formulas associated with w. By duplicating we get non-overlapping cones for different C's. In doing so we can ensure that the S_w relation will never 'exit from' a given C-critical cone.

However, for the completeness of **ILW** we don't need an **IL**-countermodel, we need an **ILW**-countermodel. The third and most difficult problem we have to deal with is in the extra condition that **ILW** imposes on the models: $R \circ S_w$ is to be conversely well-founded.

In the following definition, we isolate a special kind of critical successors. Unfortunately, at this point it is rather difficult to explain what makes them so special.

Definition 11 *Let* Γ *be a maximal* **ILW***-consistent subset of* Φ *and suppsose* $E \in \Phi$*.*

The set Δ *is an* E-critical solution *for* C_i *with respect to* $B_1 \rhd C_1, \ldots, B_n \rhd C_n$ *iff* Δ *is an* E*-critical successor of* Γ *such that* C_i *and* $\Box\neg B_1, ..., \Box\neg B_n$ *all occur together in* Δ*.*

The following lemma, which strengthens Lemma 10 above, will help us to construct the S_w-relation on a given E-critical cone in a step-by-step construction. We say that a maximal consistent set Δ *blocks* a set of formulas Ψ if Δ contains $\neg B$ for all B in Ψ.

Lemma 12 *Let Γ be a maximal consistent subset of Φ and $E \in \Phi$. Suppose $B_1 \rhd C_1, \ldots, B_n \rhd C_n$ are $\rhd$-formulae in Γ such that E admits each B_i.*

There is a non-empty subset X of $\{1, \ldots, n\}$ such that, for each $i \in X$, there exists an E-critical solution for C_i with regard to $B_1 \rhd C_1, \ldots, B_n \rhd C_n$ which blocks $\{B_j \mid j \in \{1, \ldots, n\} \backslash X\}$.

Proof. Suppose no such subset X of $\{1, \ldots, n\}$ exists. Then, in the first place, the whole set $\{1, \ldots, n\}$ does not function as an X with the required properties: There exists i such that no E-critical solution for C_i with regard to $B_1 \rhd C_1, \ldots, B_n \rhd C_n$ can be found. Without loss of generality we may assume that $i = n$.

Formally this means that there are $A_1, \ldots, A_m$ with $\Box A_1, \ldots, \Box A_m \in \Gamma$ and $F_1, \ldots, F_k$ with $F_1 \rhd E, \ldots, F_k \rhd E \in \Gamma$ such that,

$$\begin{aligned}&A_1, \ldots, A_m, \Box A_1, \ldots, \Box A_m, \neg F_1, \ldots, \neg F_k, \Box\neg F_1, \ldots, \Box\neg F_k \vdash \\ &C_n \to \Diamond B_1 \vee \ldots \vee \Diamond B_n.\end{aligned}$$

In other words,

$$\begin{aligned}&A_1, \ldots, A_m, \Box A_1, \ldots, \Box A_m \vdash \\ &C_n \to \Diamond(B_1 \vee \ldots \vee B_n) \vee (F_1 \vee \ldots \vee F_k) \vee \Diamond(F_1 \vee \ldots \vee F_k),\end{aligned}$$

which gives

$$\begin{aligned}&\Box A_1, \ldots, \Box A_m \vdash \\ &\Box(C_n \to \Diamond(B_1 \vee \ldots \vee B_n) \vee (F_1 \vee \ldots \vee F_k) \vee \Diamond(F_1 \vee \ldots \vee F_k)).\end{aligned}$$

This means that:

$$\Box A_1, \ldots, \Box A_m \vdash C_n \rhd \Diamond(B_1 \vee \ldots \vee B_n) \vee (F_1 \vee \ldots \vee F_k) \vee \Diamond(F_1 \vee \ldots \vee F_k).$$

In view of (J5) this can be simplified to

$$\Box A_1, \ldots, \Box A_m \vdash C_n \rhd \Diamond(B_1 \vee \ldots \vee B_n) \vee (F_1 \vee \ldots \vee F_k).$$

Since $B_n \rhd C_n \in \Gamma$, we see, by applying several axioms, that:

$$\Gamma \vdash B_n \rhd \Diamond(B_1 \vee \ldots \vee B_n) \vee (F_1 \vee \ldots \vee F_k).$$

At this point the axiom W plays its crucial role; we obtain:

$$\Gamma \vdash B_n \rhd (\Diamond(B_1 \vee \ldots \vee B_n) \vee (F_1 \vee \ldots \vee F_k)) \wedge \neg\Box B_n,$$

which simplifies to

$$\Gamma \vdash B_n \rhd \Diamond(B_1 \vee \ldots \vee B_{n-1}) \vee (F_1 \vee \ldots \vee F_k).$$

Since each $F_i \rhd E$ is a member of Γ this leads, by some applications of (J3) and (J2), to:

$$\Gamma \vdash B_n \rhd \Diamond(B_1 \vee \ldots \vee B_{n-1}) \vee E. \qquad (*)$$

This concludes our use of the assumption that the whole set $\{1, \ldots, n\}$ is not an X with the required properties.

Next, in the second place, the set $\{1, \ldots, n-1\}$ does not function as such an X. There exists i, say $n-1$, such that no E-critical solution for C_i with regard to $B_1 \rhd C_1, \ldots, B_n \rhd C_n$ blocking $\{B_n\}$ exists. This means that there are $A_1, \ldots, A_m$ with $\Box A_1, \ldots, \Box A_m \in \Gamma$ and $F_1, \ldots, F_k$ with $F_1 \rhd E, \ldots, F_k \rhd E \in \Gamma$, such that

$$A_1, \ldots, A_m, \Box A_1, \ldots, \Box A_m, \neg F_1, \ldots, \neg F_k, \Box\neg F_1, \ldots, \Box\neg F_k \vdash$$
$$C_{n-1} \rightarrow \Diamond(B_1 \vee \ldots \vee B_{n-1} \vee B_n) \vee B_n.$$

Reasoning as before gives

$$\Gamma \vdash B_{n-1} \rhd \Diamond(B_1 \vee \ldots \vee B_{n-1}) \vee (\Diamond B_n \vee B_n) \vee E,$$

and hence, applying (*):

$\Gamma \vdash B_{n-1} \rhd \Diamond(B_1 \vee \ldots \vee B_{n-1}) \vee E$, and thus, by W,
$\Gamma \vdash B_{n-1} \rhd \Diamond(B_1 \vee \ldots \vee B_{n-2}) \vee E$.

Continuing like this, in stage p we have

$$\Gamma \vdash B_{n-p+1} \rhd \Diamond(B_1 \vee \ldots \vee B_{n-p}) \vee E.$$

Now suppose the set $\{1, \ldots, n-p\}$ does not function as such an X: There exists i, say $n-p$, such that there are $A_1, \ldots, A_m$ with $\Box A_1, \ldots, \Box A_m \in \Gamma$ and $F_1, \ldots, F_k$ with $F_1 \rhd E, \ldots, F_k \rhd E \in \Gamma$, such that

$$A_1, \ldots, A_m, \Box A_1, \ldots, \Box A_m, \neg F_1, \ldots, \neg F_k, \Box\neg F_1, \ldots, \Box\neg F_k \vdash$$
$$C_{n-p} \rightarrow \Diamond(B_1 \vee \ldots \vee B_n) \vee B_{n-p+1} \vee \ldots \vee B_n$$

or in other words,

$$A_1, \ldots, A_m, \Box A_1, \ldots, \Box A_m, \neg F_1, \ldots, \neg F_k, \Box\neg F_1, \ldots, \Box\neg F_k \vdash$$
$$C_{n-p} \rightarrow \Diamond(B_1 \vee \ldots \vee B_{n-p}) \vee (\Diamond B_{n-p+1} \vee B_{n-p+1}) \vee \ldots \vee (\Diamond B_n \vee B_n)$$

Reasoning as before and applying the results reached for $B_n, \ldots, B_{n-p+1}$, we find

$$\Gamma \vdash B_{n-p} \rhd \Diamond(B_1 \vee \ldots \vee B_{n-p-1}) \vee E.$$

Continuing like this, we finally get, in the n-th stage,

$$\Gamma \vdash B_1 \rhd E,$$

but this is a contradiction, since $B_1 \rhd E$ cannot be a member of Γ. $\dashv$

Theorem 13 (*Completeness and decidability of* **ILW**). *If $\nvdash_{\mathbf{ILW}} A$, then there is a finite* **ILW***-model $\langle W, R, S, \Vdash \rangle$ such that $w \nVdash A$ for some $w \in W$.*

Proof. We show that, for abitrary Φ, Γ with Φ adequate and Γ maximal consistent in Φ, there is a model with root w_0 such that, for all $\phi \in \Phi$, $w_0 \Vdash \phi$ iff $\phi \in \Gamma$. Then for completeness it is sufficient to take some finite adequate set Φ containing $\neg A$ and let Γ be a maximal consistent subset of Φ containing $\neg A$.

Every world in the model will be a sequence of pairs $\langle\langle\Delta_1, \sigma_1\rangle, \ldots, \langle\Delta_k, \sigma_k\rangle\rangle$. In this sequence each Δ_i is a maximal consistent subset of Φ and each σ_i is either empty or a pair consisting of a formula in Φ and a number j $(1 \leq j \leq n)$, where n is the number of $\rhd$-formulas in Φ. If w is such a world, Δ_k will be the set of the formulas true in the world, and so we will write $\Delta(w)$ for Δ_k. The sequence that codes the world encrypts the sequence of all its predecessors as its initial segments. The formula E, if any, in σ_k signals that w is in the E-critical cone of its immediate predecessor w'. The natural number accompanying E is used to fix the $S_{w'}$-relation inside this E-critical cone.

More precisely, the set of worlds W of the model is given by the following inductive definition.

(i) $w_0 = \langle\langle\Gamma, \emptyset\rangle\rangle \in W$,

(ii) If $\tau * \langle\Delta, \sigma\rangle \in W$, the following procedure is applied for each E that occurs as a consequent in some $\rhd$-formula in Φ. Taking the set $Y = \{B_1 \rhd C_1, \ldots, B_n \rhd C_n\}$ of all $\rhd$-formulas in Δ with antecedents admitted by E as our starting point, we repeatedly apply lemma 12 with respect to Δ and E in Φ. Eventually we obtain a sequence $X_1, \ldots, X_m$ (obtained in that order) of disjoint subsets of Y the union of which is Y.

The set X_1 is the outcome of applying lemma 12 to Δ, E and Y. Choose for each $j \in X_1$ an E-critical solution Δ_j for C_j with respect to Y blocking $\{B_j \mid j \in Y \backslash X_1\}$. Then extend W with $\tau * \langle\Delta, \sigma\rangle * \langle\Delta_j, \langle E, 1\rangle\rangle$.

In a similar manner X_{i+1} is determined: X_{i+1} is the set obtained by applying lemma 12 to $Y \backslash (X_1 \cup \ldots \cup X_i)$. Choose for each $j \in X_{i+1}$ an E-critical solution Δ_j for C_j with respect to $Y \backslash (X_1 \cup \ldots \cup X_i)$ containing $\neg B_m$ for all $m \in Y \backslash (X_1 \cup \ldots \cup X_{i+1})$. Then extend W with $\tau * \langle\Delta, \sigma\rangle * \langle\Delta_j, \langle E, i+1\rangle\rangle$.

W is finite for the usual reasons: each newly constructed world 'contains' more $\Box$-formulas than its immediate predecessor.

Define R and S_w on W as follows:

(i) $w\,R\,w'$ iff w is a proper initial segment of w'.

(ii) $u\,S_w v$ iff $u = w * \langle\Delta, \langle E, i\rangle\rangle * \sigma$ and $v = w * \langle\Delta', \langle E, j\rangle\rangle * \tau$ and (either $j < i$, or $j = i$ and σ is empty), or $u = v$ or $u\,R\,v$.

Note that $\{u \mid u = w * \langle \Delta, \langle E, i \rangle \rangle * \sigma$ for some $\Delta, i, \sigma\}$ plays the role of the E-critical cone above w. For each i, we call the set $\{u \mid u = w * \langle \Delta, \langle E, i \rangle \rangle * \sigma\}$ the i-th section of this cone.

Given its definition, it is obvious that R is transitive and conversely well-founded. Likewise, S_w is easily seen to be reflexive and transitive. For the converse well-foundedness of $R \circ S_w$ it is sufficient to note that, if $u \, R \circ S_w \, v$, then, either $u \, R \, v$ or v's 'index' (i.e. the number accompanying the last element of v) is lower than the index of u.

Finally, it remains to prove that for all $B \in \Phi, w \in W_\Gamma$, $w \Vdash B$ iff $B \in \Delta(w)$. The induction is trivial except for the two $\rhd$-cases.

First assume $\neg(C \rhd D) \in \Delta(w)$. This is easy: D admits C, so some D-critical successor with C in it exists, because C will occur as C_i in one of the X_j's, and its D-critical solution will be produced in the D-critical cone above w. By the definition above the S_w-relation does not exit from the D-critical cone.

Next assume $C \rhd D \in \Delta(w)$. We have to show that, if C occurs in the E-critical cone above w, then so does D in such a way that the occurrence of D can be reached from the occurrence of C by S_w. Since $C \rhd D \in \Delta(w)$, $C \rhd D$ is one of the $B_i \rhd C_i$ in Y. This number i is an element of X_j for some j while an E-critical solution for C_i is produced as the formula set of a world v in the j-th section of the E critical cone above w, in fact as one of the R-minimal elements of that section. For all $k \leq j$, $\Box\neg B_i$ is present in the R-minimal elements of the k-sections of the E-critical cone above w. This is so because 'before' X_j, $B_i \rhd C_i$ is each time a member of the set of formulas under discussion. This implies that, in these k-sections, B_i does not occur in the nonminimal elements. Moreover, for $k < j$, B_i does not occur among the minimal elements of the k-section either. Therefore, if B_i occurs in the E-critical cone at all, it will be either in a world u belonging to a k-section with $k > j$, or in a world u that is a minimal element of the j-section. In both cases $u \, S_w \, v$ holds, so $C \rhd D$ is forced in w. $\dashv$

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