Determiners, Adjectives and a Query of van Benthem's

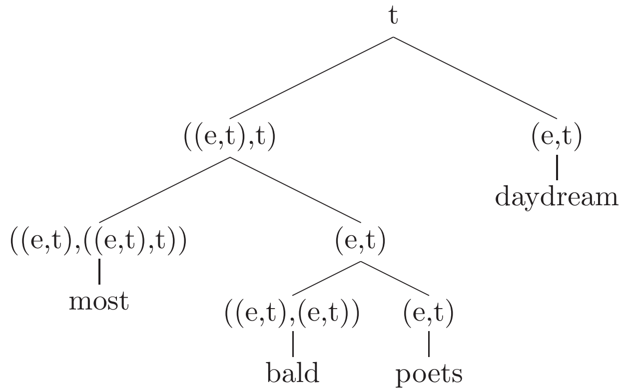
Edward L. Keenan

March, 1999

In this note I provide an answer to an apparently technical query by van Benthem [4] concerning denotations of English expressions. The answer turns out to be revealing of some systematic semantic differences associated with certain categories of expression. The categories of interest to us are illustrated in (1a) and given an extensional type theoretic analysis in (1b).

(1) a. Most bald poets daydream

b.



So we interpret sentences (Ss) as elements of the boolean algebra $\mathbf{2} = \{1, 0\}$. And for each universe E of objects, we interpret nouns (Ns), such as *poet,* and verb phrases (VPs) such as *daydream,* as functions from E into $\mathbf{2}$. We call such functions (extensional) *properties* (over E). We write P_E for the set of these properties; in general $[A \to B]$ is the set of functions from A into B. So $P_E = [E \to \mathbf{2}]$. As P_E is isomorphic to $\wp(E)$, the power set of E, we often treat properties as subsets of E. (The map sending each f in $[E \to \mathbf{2}]$ to $\{a \in E \mid f(a) = 1\}$ is an isomorphism). Adjectives, e.g. *bald,* combine with expressions of type (e,t) to form ones of that same type, and thus denote in $[P_E \to P_E]$. NPs (Noun Phrases) combine with VPs to form Ss and have type ((e,t),t), denoting in $[P_E \to \mathbf{2}]$. Elements of this set are called *generalized quantifiers* (over E). And Determiners (Dets), such as *most, all, no, more than ten,* $\ldots$, combine with Ns to form NPs and thus denote in $[P_E \to [P_E \to \mathbf{2}]]$.

Leading up to van Benthem's query, we note that the denotation sets we have presented—$\mathbf{2}$, P_E, $[P_E \to \mathbf{2}]$, $[P_E \to [P_E \to \mathbf{2}]]$, and $[P_E \to P_E]$—possess a natural boolean structure. For $\mathbf{2}$ this structu is given by the standard truth tables for conjunction, disjunction and negation. For the generalized quantifiers, $[P_E \to \mathbf{2}]$, it is given *pointwise,* illustrated by:

(2) a. Every boy and some girl daydreamed $\leftrightarrow$ Every boy daydreamed and some girl daydreamed

$$(F \wedge G)(p) = F(p) \wedge G(p)$$

b. Every boy or some girl daydreamed $\leftrightarrow$ Every boy daydreamed or some girl daydreamed

$$(F \vee G)(p) = F(p) \vee G(p)$$

c. Not every boy daydreamed $\leftrightarrow$ It is not the case that every boy daydreames

$$(\neg F)(p) = \neg(F(p))$$

More generally, given a set B with a boolean structure and any non-empty set A, $[A \to B]$ has a boolean structure given pointwise by:

(3) a. $(F \wedge G)(a) = F(a) \wedge G(a)$

b. $(F \vee G)(a) = F(a) \vee G(a)$

c. $(\neg F)(a) = \neg(F(a))$

Here of course $\wedge$ is the interpretation of 'and', $\vee$ that of 'or', and $\neg$ that of 'not'. Also, given that $[P_E \to \mathbf{2}]$ has a boolean structure we observe that $[P_E \to [P_E \to \mathbf{2}]$ also has a boolean structure pointwise, providing the correct interpretation for conjunctions, disjunctions, and negations of Dets. E.g.

(4) (most but not all)(students) $\leftrightarrow$ most students but (not all) students

$\leftrightarrow$ most students but not (all students)

Similarly the boolean structure of $[E \to \mathbf{2}]$ is given pointwise, and corresponds to intersection, union, and complement (relative to E) on its power set construal.

Observe now that a generalized quantifier is a function from one boolean structure, P_E, to another, $\mathbf{2}$. There is a subset of these functions which has a distinctive, indeed determinative, role in the set as a whole. These functions are denotable by proper nouns: *Dana, Robin,* . . . . (5) illustrates one of their distinctive properties.

(5) a. Dana both lies and cheats $\leftrightarrow$ Dana lies and Dana cheats

b. Dana either lies or cheats $\leftrightarrow$ Either Dana lies or Dana cheats

c. Dana doesn't smoke $\leftrightarrow$ It is not the case that Dana smokes

Call the functions denoted by proper nouns *individuals*: I, I', I'', ... Using p and q as variables ranging over extensional properties, we render the semantic equivalences above by:

(6) a. $I(p \wedge q) = I(p) \wedge I(q)$

b. $I(p \vee q) = I(p) \vee I(q)$

c. $I(\neg p) = \neg(I(p))$

In other words, individuals are boolean homomorphisms. Moreover [1] every (complete[1]) boolean homomorphism f m P_E into $\mathbf{2}$ is an individual. Thus proper

[1]'complete' just means that arbitrary greatest lower bounds and least upper bounds are preserved: $h(\bigwedge_i p_i) = \bigwedge_i h(p_i)$ and dually for joins: $h(\bigvee_i p_i) \bigvee_i h(p_i)$. 'homomorphism' and 'complete homomorphism' coincide for finite sets $\{p_i \mid i \in I\}$, but for infinite such sets there will be homomorphisms which are not complete.

noun denotations are semantically characterized as the (complete) boolean homomorphisms from P_E into $\mathbf{2}$.

Now van Benthem observes that the denotation sets for Dets, $[P_E \rightarrow [P_E \rightarrow \mathbf{2}]]$, are also maps from one boolean set to another, yet we do not find linguistically natural classes of Dets which denote boolean homomorphisms. Another example, not considered by him, are the Adjectives: $[P_E \rightarrow P_E]$. van Benthem's query is whether this semantic distribution accidental, or is there some reason to expect homomorphims among the NPs but not among the Dets or Adjectives?

We consider the negative case first: Why are there no, or few, Dets and Adjectives which must be interpreted as homomorphisms? The answer we give is that independent constraints on the denotations of Dets and Adjs rule out almost all homomorphisms. So few expressions can denote homomorphisms since there are few homomorphisms to denote.

We consider Adjectives first. Extensional Adjectives[2] denote in $[P_E \rightarrow P_E]$. But, as per Keenan & Faltz [2] extensional Adjs are always restricting. To say that an Adj *blik* is restricting is to say that for any property p, the *blik* p's are a subset of the p's; that is, every *blik* p is a p. So bald is restricting since, e.g. a bald artist is necessarily an artist. *tall* is restricting since, e.g. every tall linguist is a linguist; and skillful is restricting: all skillful surgeons are surgeons.[3]

Many functions from P_E to P_E are not restricting. For example let F be that map from P_E to P_E which sends each property p to $E-p$. Clearly $F(p) \nsubseteq p$ for any proper subset p of E, so F is not restricting. A less dramatic case would be a function G mapping each p to $p \cup \{b\}$, b a fixed element of E. So e.g. $G(\emptyset) = \{b\} \nsubseteq \emptyset$, so G is not restricting.

Now observe that the empirically motivated condition of being restricting prevents[4] Adjective denotations from being homomorphisms, with precisely one

[2]To say that an Adjective *blik* is extensional is to say (by way of example) that in any situations in which it happens that t surgeons and the lawyers are the same we can infer that the blik surgeons and the blik lawyers are the same. *bald* is clearly extensional; so is *tall,* so are many relative clauses, such as *who Bill saw*: If the surgeons and the lawyers happen to be the same individuals in some situation s then the surgeons who Bill saw must be the same individuals as the lawyers who Bill saw in that situation. *skillful,* and value judgment adjectives in general (*good, bad, accomplished, exceptional, terrible, . . .*) are not. John might be both a surgeon and a lawyer but a skillful lawyer but not a skillful surgeon.

[3]The more general definition is as follows: For $(B, \leq)$ a partially ordered set, a function f from B into B is *restricting* iff $f(x) \leq x$, all $x \in B$. To say that B is a boolean structure is to say that $(B, \leq)$ is a (complete) *boolean lattice,* meaning that the $\leq$ relation is a partial order (reflexive, antisymmetric and transitive) which meets several other conditions. For example, each subset K of B has a *greatest lower bound* (glb), that is, an element x such that (i) for all $k \in K, x \leq k$; that is, x is a *lower bound* (lb) for K, and (ii) for every lb y for $K, y \leq x$. That is, x is greatest of the lower bounds. We write $\bigwedge K$ for the glb of K, and when $K = \{x, y\}$ we usually write $x \wedge y$ instead of $\bigwedge\{x, y\}$. Dually we write $\bigvee K$ for the *least upper bound* for K, using $x \vee y$ for $\bigvee\{x, y\}$.

Note that many common adjectives that are not extensional are nonetheless restricting: a skillful surgeon is a surgeon. But some, such as *apparent* and *alleged,* are not.

[4]It also entails that the Adjective algebra $[P_E \rightarrow P_E]$ is not a pointwise one. In fact greatest lower bounds and least upper bounds do behave pointwise, but negation does not. To be restricting, $\neg f$, the complement of a restricting function f, cannot simply map each p to $\neg(f(p))$, as that would in general fail to be a subset of p. Rather $\neg f$ must map each p to

exception.

Theorem 1 *The only function from a boolean set P to itself which is both restricting and preserves complements is the identity map. So the only restricting homomorphism is the identity function.*

Proof. First the identity map $\mathrm{id} : B \to B$ is clearly restricting: $\mathrm{id}(p) \le p$ since $\mathrm{id}(p) = p$. And id is a hom: $\mathrm{id}(p \wedge q) = p \wedge q = \mathrm{id}(p) \wedge \mathrm{id}(q)$, and similarly for the other boolean operations. In the other direction, let h be a restricting homomorphism from P into P. Let p arbitrary in P. Then $h(p) \le p$ since h is restricting. And because h is restricting $h(\neg p) \le \neg p$, so $p \le \neg(h(\neg p)) = h(\neg\neg p)$, h preserves complements, $= h(p)$. Thus $h(p) = p$, and since p was arbitrary, $h = \mathrm{id}$. □

Thus we see that with the exception of a single logical constant, the distinctive property of Adjectives, that of being restricting, is incompatible with being a homomorphism, indeed incompatible with just preserving complements.

A comparable, though not quite as strong a claim, obtains for the Determiners. Here extensional Dets quite generally are conservative, defined by:

Definition 1 *a function D from P_E into $[P_E \to \mathbf{2}]$ is conservative (CONS) iff for all properties p, q*

(7) $$D(p)(q) = D(p)(p \cap q).$$

So we see that MOST is CONS since, e.g. *Most students are vegetarians* must have the same truth value as *Most students are both students and vegetarians.* By contrast a function D such that $D(p)(q) = 1$ iff the number of individuals with p is the same as the number with q is not conservative. Imagine a universe E in which there is just one boy and one girl and no one is both a boy and a girl. Then $D(\mathrm{BOY})(\mathrm{GIRL})$ is true since the number of boys and the number of girls is the same, 1. But since the number of boys, 1, is not the same as the number of boys who are girls, 0, $D(\mathrm{BOY})(\mathrm{BOY} \cap \mathrm{GIRL}) = 0$, so D is not CONS.

Now we show that CONS rules out all but a degenerate class of homomorphisms. The argument is partially similar to that for adjectives, and indeed there is a non-trivial similarity between being restricting as an Adj and conservative as a Det: The value of a restricting adjective f at a property p is given by a single subproperty s of p, namely $s = f(p)$. (A subproperty of p is a property s with $s \subseteq p$). And where D is a (conservative) Det function, its value at a property p is given by a set of subproperties s of p, namely those subproperties s such that $D(p)(s) = 1$. This determines the value of $D(p)$ at any q, since $D(p)(q)$ is the same as $D(p)(p \cap q)$, and the property $(p \cap q)$ is a subproperty of p. So Adj+N is, semantically, a subproperty of t N denotation; Det+N is a

$p \cap \neg(f(p))$. Linguistically this just says that the not very tall students are not simply the objects that fail to be very tall students—my car, or anything which isn't a student has that property; rather they must be the students who fail to be very tall students.

set of subproperties of the N denotation. In this sense a Det is something like a "higher order" adjective.

Let us say first that a function F from a set A to a set B is *uninformative* if F is constant: that is, for all $x, y \in A, F(x) = F(y)$. An expression will be called *uninformative* if it always denotes an uninformative function. For example, an NP like *fewer than zero cats* is uninformative, since no matter what property q we pick, it cannot be the case the number of cats with q is less than zero. So the denotation of *fewer than zero cats* is constant, mapping all properties to 0.

Note that with boolean compounds in *and, or,* and *not* we can always form uninformative expressions. For example, if we took Ss as denoting functions from "possible worlds" to truth values then an S of the form "P and it is not the case that P" is uninformative, mapping all possible worlds w to 0. Similarly "P or it is not the case that P" is uninformative, mapping all possible worlds w to 1. Similarly an NP like *either more than ten cats or else at most ten cats* is uninformative since its denotation maps each property q to 1. We write 1_{GQ} for that GQ mapping all properties to 1; 0_{GQ} maps all properties to 0.

Now, while we can build syntactically complex NPs that are uninformative, we do not expect to find lexical (= monomorphemic) NPs that are uninformative. What communicative purpose would be served by having a class of lexical NPs each of which was true (false) of all propereties? For such an NP, the combination NP+Predicate would never vary in truth value and thus could not be used to distinguish the way our world is from any other possible world. Similar claims hold for Dets: what communicative value would be served by a lexical Det *blik* with the property that for any noun N, blik+N was uninformative?

Now we will show that the only homomorphic Dets are "degenerate" in the sense that the NPs they build are always uninformative. First,

Lemma 1 *Let $D \in [P_E \rightarrow [P_E \rightarrow \mathbf{2}]]$ be conservative and respect complements. Then for all $p \in P_E$, $D(p)$ is uninformative. Thus any conservative (complete) homomorphism is degenerate in this sense.*

Proof. Let D arbitrary as above, let $p \in P_E$. We show that $D(p)$ is constant. Let $q \in P_E$ be arbitrary. Then,

$$\begin{array}{rcll}
D(p)(q) & = & D(p)(p \cap q) & D \text{ is CONS} \\
& = & D(\neg\neg p)(p \cap q) & \text{double complements} \\
& = & (\neg(D(\neg p))(p \cap q) & D \text{ respects } \neg \\
& = & \neg(D(\neg p)(p \cap q)) & \text{pointwise } \neg \\
& = & \neg(D(\neg p)(\neg p \cap p \cap q)) & D \text{ is CONS} \\
& = & \neg(D(\neg p)(\emptyset)) & \text{set theory} \\
& = & (\neg(D(\neg p))(\emptyset) & \text{pointwise } \neg \\
& = & D(\neg\neg p)(\emptyset) & D \text{ respects } \neg \\
& = & D(p)(\emptyset) & \text{double complements}
\end{array}$$

Since q was arbitrary we infer that for all q and q', $D(p)(q) = D(p)(\emptyset) = D(p)(q')$, so $D(p)$ is constant. □

Lemma 1 suffices to show that we do not expect natural classes of Determiners to be homomorphisms, indeed, as with adjectives, we do not expect them to even respect complements. If they did, they would always determine uninformative NPs.

Can we do better than this? That is, are there actually any conservative functions that do build uninformative GQs? The answer is affirmative.

Lemma 2 *Let D be conservative and a complete homomorphism. Then,*

1. *For some $b \in E, D(\{b\}) = 1_{GQ}$. Since homomorphisms map unit elements to unit elements, $D(E) = 1_{GQ}$. And since D is a complete hom, $D(E) = D(\bigcup_{a \in E}\{a\} = \bigvee_{a\in E} D(\{a\})$. This last would be 0_{GQ} if D didn't map any $\{b\}$ to 1_{GQ}, since then lemma 1 it would map all $\{b\}$ to 0_{GQ}, contradicting that $D(E) = 1_{GQ}$.*

2. *by 2.1 let b such that $D(\{b\}) = 1$. Since homs are increasing, then for all p with $b \in p$, $D(p) = 1_{GQ}$. Now let $b' \in E$ be different from b. If $D(\{b'\}) = 1_{GQ}$ then $D(\{b\}\cap\{b'\}) = D(\{b\})\wedge D(\{b'\}) = 1_{GQ}\wedge 1_{GQ} = 1_{GQ}$. But this is a contradiction as $D(\{b\}\cap\{b'\}) = D(\emptyset) = 0_{GQ}$, since all homs map zero elements to zero elements.*

3. *$D(p) = 0_{GQ}$ for all p such that $b \notin p$. This is so since*
$$D(p) = \bigvee_{b' \in p} D(\{b'\}) = \bigvee\{0_{GQ}\} = 0_{GQ}.$$
Thus we see that if D is a conservative complete homomorphism then for some $b \in E$, $D(p)(q) = 1$ iff $b \in p$, all $p, q \subseteq E$.

4. *for each $b \in E$ define $D_b \in [P_G \to [P_E \to \mathbf{2}]]$ as follows:*
$$D_b(p) = \begin{cases} 1_{GQ} & \textit{if } b \in p \\ 0_{GQ} & \textit{if } b \notin p \end{cases}$$
Then D_b is a conservative complete homomorphism. Clearly D_b is CONS: $D_b(p)(q) = D(p)(p\cap q)$ since $D_b(p)$ is constant. To see that D_b is a c-hom observe:
$$\begin{aligned} D_b(\bigcap_i p_i) = 1 \quad & \textit{iff} \quad b \in \bigcap_i p_i && \textit{Def } D_b \\ & \textit{iff} \quad \textit{for all } i, b \in p_i && \textit{Def } \bigcap \\ & \textit{iff} \quad \textit{for all } i, D_b(p_i) = 1 && \textit{Def } D_b \\ & \textit{iff} \quad \bigwedge_i D_b(p_i) = 1 && \textit{Def } \bigwedge \end{aligned}$$
Thus $D_b(\bigcap_i p_i) = \bigwedge_i D_b(p_i)$, so D_b preserves arbitrary greatest lower bounds. Similarly D_b preserves complements, hence D_b is a complete homomorphism.

Lemma 2 guarantees that the complete homomorphisms among the Det denotations are precisely the functions D_b, for $b \in E$.

And it is unsurprising that English does not present a natural class of Dets which are homomorphism denoting, as the independent constraints on Det denotations guarantee that the only functions they could denote would be degenerate in the sense of always yielding uninformative generalized quantifiers.

We conclude with the last question: is there any reason why we might expect to find (complete) homomorphism denoting expressions among the NP denotations? And again we have an affirmative answer, though of a qualitatively different sort.

Observe first that we think of VP denotations as predicating properties of elements of E. Traditionally a simple subject-predicate S like *Jo daydreams* is true iff the object denoted by *Jo* has the property denoted by *daydreams.* So we think of proper nouns, like *Jo* (and deictically interpreted pronouns like *he* and *she*), as denoting elements of E.

But this approach is insufficient to handle the interpretation of complex NPs, indeed even of just elementary boolean compounds of proper nouns themselves. The following example illustrates this. Imagine a situation in which E has just two elements, j and p, denoted by *Jo* and *Pat* respectively. Suppose further that in this model only Jo daydreams and only Pat works hard. Now the boolean compound *both Jo and Pat* can't denote j, since then the S *Both Jo and Pat daydream* would be true, and it isn't, since it entails that Pat daydreams, which is false in this model. But equally it can't denote p, since then *Both Jo and Pat* work hard would be true, and it isn't. And this exhausts the elements of E, so the NP *both Jo and Pat* lacks a denotation in E.

Building on this example we see that the number of logically distinct NPs built as boolean compounds (in *and, or, not, neither ... nor ...*) of proper nouns is the same as the number of sets of properties. First, for each NP X in (8) there is exactly one property q_X which is such that "X daydream(s)" is true when *daydream(s)* is interpreted as q. Moreover the q's are different for different NPs.

(8)

X	q_X
both Jo and Pat	$\{j, p\}$
Jo but not Pat	$\{j\}$
Pat but not Jo	$\{p\}$
neither Jo nor Pat	$\emptyset$

So each of the four properties can be uniquely identified by an NP. But now consider disjunctions of these NPs. The S *Either Jo but not Pat or Pat but not Jo daydreams* is true when *daydream* is interpreted as either $\{j\}$ or as $\{p\}$. And in general each subset of the NPs X above determines the set of properties determined by each disjunct. So different subsets yield different property sets. There are 16 distinct subsets of 4 NPs and thus 16 logically distinct NPs built just by forming boolean compounds of proper nouns. And in general the set of logically distinct boolean compounds of individual denoting NPs has cardinality 2^k, where k is the number of extensionally distinct VPs, itself of cardinality $2^{|E|}$.

Thus closing the NPs under boolean compounds of proper nouns forces us to consider the denotation set of NPs to be, up to isomorphism, the full set of functions from VP denotations into S denotations, that is, the set $[P_E \to \mathbf{2}]$ of generalized quantifiers. But we still maintain our original intuition that proper noun denotations correspond to elements of E. That is, we want *Jo daydreams* to be true iff a fixed element of j of E has the property of daydreaming, that is, $\mathrm{DAYDREAM}(j) = 1$. This then forces proper noun denotations to be *individuals* as defined in (9), whence it follows that proper noun denotations are complete homomorphisms (for proof see e.g. [1]).

(9) For all $b \in E$, define $I_b \in [P_E \to \mathbf{2}]$ by: $I_b(p) = p(b)$

Theorem 2 *For all $F \in [P_E \to \mathbf{2}]$, F is a complete homomorphism iff for some $b \in E$, $F = I_b$.*

Note that in categorial grammar terms I_b is the result of lifting ("raising") b to be a function from $[[E \to \mathbf{2}] \to \mathbf{2}]$. More generally,

(10) For B a (complete) boolean algebra and A any set, we define for each $a \in A$, $\mathrm{Lift}_B(a)$ in $[[A \to B] \to B]$ by setting $\mathrm{Lift}_B(a)(F) = F(a)$.

Theorem 3 *$\mathrm{Lift}_B(a)$ is a complete homomorphism (where $[A \to B]$ is understood pointwise).*

Proof. The proof is by direct computation. E.g. $\mathrm{Lift}_B(a)(F \wedge G) = (F \wedge G)(a) = F(a) \wedge G(a) = \mathrm{Lift}_B(a)(F) \wedge \mathrm{Lift}_B(a)(G)$. Similarly $\mathrm{Lift}_B(a)$ respects complements. $\square$

Thus we find homomorphisms among NPs as a consequence of raising elements of the universe to be functions on one place predicate denotations. It remains only to emphasize that the raising is necessary in that for finite E, all functions from one place predicate denotations to truth values are denotable [3]. Indeed the set of generalized quantifier $[P_E \to \mathbf{2}]$ is precisely the complete boolean closure of the individuals in this set, that is, the set of possible proper noun denotations.

References

[1] Edward L. Keenan. Natural language, sortal reducibility and generalized quantifiers. *The Journal of Symbolic Logic*, 58(1):314–325, 1993.

[2] Edward L. Keenan and Leonard M. Faltz. *Boolean Semantics for Natural Language*. D. Reidel, 1985.

[3] Edward L. Keenan and Jonathan Stavi. A semantic characterization of natural language determiners. *Linguistics and Philosophy*, 9:253–326, 1986.

[4] Johan van Benthem. *Essays in Logical Semantics*. D. Reidel, 1986.