D-Structures and their Semantics

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Abstract

In these notes we shall be concerned with a semantic object which is a generalization of classical structures, Kripke structures and the regular $*$-structures of Ehrenfeucht-de Jongh. We shall start by showing how these different cases can be obtained by imposing different regularly conditions on the basic object (D-structures) and the semantics can then be directly interpreted into the semantics of D-structures. We shall then give a game-theoretic explanation of the semantics of the D-structures from which the finite model property of regular $*$-sructures can be easily obtained. We go on to look at the proof theory of these objects.

Contents

1 Introduction

In this survey we shall show that a D-structure is a very flexible (but nontrivial) type of object and includes classical structures, intutionistic structures[1], and the regular $*$-structures of Ehrenfeucht-de Jongh as special cases The intuitive idea is this. At any moment of time, what we know about the world is a finite amount, but as time passes, and if our memory is good, this finite amount increases. In other words, the old information is embedded in the new information. The *way* in which the old information is embedded can be chosen in various possible ways and these different choices lead to different semantics.

In the following, μ will be a finite relational type. Constants are permitte but not function symbols.

Definition 1 A D-*structure* $\mathcal{M}$ of type μ consists of two objects:

1. a family $\mathcal{F}$ of finite relational structures (diagrams), all of type μ and
2. a family $\mathcal{H}$ of homomorphisms between elements of $\mathcal{F}$. $\mathcal{H}$ includes all the identity maps. $\mathcal{H}^t$ is the closure of $\mathcal{H}$ under composition and clearly $\langle \mathcal{F}, \mathcal{H}^t \rangle$ will be a category.

Remark: Note that homomorphisms preserve atomic formulae but not necessarily their negations. Members of $\mathcal{H}$ will be called *allowable maps*.

Definition 2 A D-structure $\mathcal{M}$ will be said to be *rigid* if all allowable maps are inclusions. It is *directed* if given D_1, D_2 in $\mathcal{F}$ there is a D_3 and allowable maps $p_1 : D_1 \to D_3$ and $p_2 : D_2 \to D_3$. $\mathcal{M}$ is *weakly* directed if $\langle \mathcal{F}, \mathcal{H}^t \rangle$ is directed.

Definition 3 Let A be a sentence of the language $\mathcal{L}_u$ augmented by constants from a diagram D (we shall take the elements themselves to be these constants) and modal operators $\Box$ and $\diamond$. We recall that $\Box$ means "necessarily" and $\diamond$ means "possibly". We define $\mathcal{M}, D \models A$ by induction on the complexity $c(A)$ of A.

1. $c(A) = 0$. Then $\mathcal{M}, D \models A$ iff A is true in D.
2. $A = B \wedge C$. Then $\mathcal{M}, D \models B \wedge C$ iff $\mathcal{M}, D \models B$ and $\mathcal{M}, D \models C$.
3. $A = B \vee C$. Then $\mathcal{M}, D \models B \vee C$ iff $\mathcal{M}, D \models B$ or $\mathcal{M}, D \models C$.
4. $A = \neg B$. Then $\mathcal{M}, D \models \neg B$ iff $\mathcal{M}, D \not\models B$.
5. $A = (\exists x) B(x)$. Then
   $\mathcal{M}, D \models (\exists x) B(x)$ iff there exists $a \in |D|$ such that $\mathcal{M}, D \models B(a)$.
6. $A = (\forall x) B(x)$. Then
   $\mathcal{M}, D \models (\forall x) B(x)$ iff for all $a \in |D|$, $\mathcal{M}, D' \models B(a)$.
7. $A = \Box B(a_1, \ldots, a_k)$. Then
   $\mathcal{M}, D \models \Box B(a_1, \ldots, a_k)$ iff for all allowable $f : D \to D'$,
   $\mathcal{M}, D' \models B(f(a_1), \ldots, f(a_k))$.

[1] The finiteness requirement on elements of $\mathcal{M}$ has to be dropped in this case, for technical reasons on the diagrams.

8. $A = \diamond B(a_1, \ldots, a_k)$. Then

$$\mathcal{M}, D \models \diamond B(a_1, \ldots, a_k) \quad \text{iff} \quad \text{for some allowable } f : D \to D', \; \mathcal{M}, D \models B(f(a_1), \ldots, f(a_k)).$$

In 7, 8 the constants from $|D|$ are explicitly displayed.

Before studying D-structures in general we shall verify the claim made on before Definition 1.

Definition 4 Let A be a formula of the language $\mathcal{L}_{\mu * D}$, i.e. $\mathcal{L}_\mu$ with constants from $|D|$. A^c is the formula obtained from A if we replace $\exists$ everywhere by $\diamond\exists$ and $\forall$ everywhere by $\Box\forall$.

Theorem 5 *Let $\mathcal{A}$ be a classical μ-structure. $\mathcal{M}^c(\mathcal{A}) = \mathcal{M}$ is th D-structure where $\mathcal{F}$ consists of all finite substructures of $\mathcal{A}$. $\mathcal{H}$ consists of all inclusion maps. (Thus $\mathcal{M}$ is directed and rigid.) A is any sentence of $\mathcal{L}_{\mu * D}$. Then*

$$\mathcal{A} \models A \qquad \textit{iff} \qquad \mathcal{M}, D \models A^c,$$

where D contains all constants of A.

Proof. $\neg$, $\vee$, $\wedge$ and atomic sentences are trivial. Suppose now that A is $(\exists x) B(x, a_1, \ldots, a_k)$ then A^c is $\diamond(\exists x) B^c(x, a_1, \ldots, a_k)$.

[left to right] Suppose $\mathcal{A} \models A$. Then there is an $a \in |\mathcal{A}|$ such that

$$\mathcal{A} \models B(a, a_1, \ldots, a_k).$$

Let D' be a substructure containing D and a. Then by induction hypothesis, $\mathcal{M}, D' \models B^c(a, a_1, \ldots, a_k)$ hence $\mathcal{M}, D' \models (\exists x) B^c(x, a_1, \ldots, a_k)$ hence

$$\mathcal{M}, D \models \diamond (\exists x) B^c(x, a_1, \ldots, a_k).$$

I.e. $\mathcal{M}, D \models A^c$.

[right to left] Suppose

$$\mathcal{M}, D \models \diamond (\exists x) B^c(x, a_1, \ldots, a_k)$$

then there is a D' such that $D \subseteq D'$ and $a \in D'$ such that $\mathcal{M}, D' \models B^c(a, a_1, \ldots, a_k)$. But then $\mathcal{A} \models B(a, a_1, a_2, \ldots, a_k)$ and hence

$$\mathcal{A} \models (\exists x) B(x, a_1, a_2, \ldots, a_k).$$

The $\forall$ case is similar. ∎

Theorem 6 *Let $\mathcal{M}$ be a directed, rigid D-structure. Let*

$$\mathcal{A} = \bigcup_{D_a \in \mathcal{F}} D_a.$$

(This union makes sense since $\mathcal{M}$ is directed and rigid.) Then, for sentences A of $\mathcal{L}_{u * \mathcal{A}}$, we have if D contains all constants of A,

$$\mathcal{M}, D \models A^c \qquad \textit{iff} \qquad \mathcal{A} \models A.$$

Proof. Quite similar to above. ∎

Definition 7 Let A be a formula of the intuitionistic predicate calculus with symbols from μ and additional constants. We define A^{i} by induction on $c(A)$.

1. $c(A) = 0$, $A^{\mathrm{i}} = A$
2. $A = B \wedge C$, $A^{\mathrm{i}} = B^{\mathrm{i}} \wedge C^{\mathrm{i}}$
3. $A = B \vee C$, $A^{\mathrm{i}} = B^{\mathrm{i}} \vee C^{\mathrm{i}}$
4. $A = \neg B$, $A^{\mathrm{i}} = \Box\neg B^{\mathrm{i}}$
5. $A = B \to C$, $A^{\mathrm{i}} = \Box(B^{\mathrm{i}} \to C^{\mathrm{i}})$
6. $A = (\forall x)B(x)$, $A^{\mathrm{i}} = \Box(\forall x)B^{\mathrm{i}}(x)$
7. $A = (\exists x)B(x)$, $A^{\mathrm{i}} = (\exists x)B^{\mathrm{i}}(x)$.

(In cases 2,3,7, we could take $A^{\mathrm{i}} = \Box(\exists x)B^{\mathrm{i}}(x)$ etc. and the next theorem will still hold.)

Definition 8 Let $\mathcal{A}$ be an intuitionistic structure (as in [Fit69] p.46). Let D_Γ be the structure with base set $P(\Gamma)$, and in which precisely those atomic A hold where $\Gamma \models A$. There is a homomorphism (which comes from set inclusion) from D_Γ to $D_{\Gamma'}$ just in case $R(\Gamma, \Gamma')$. Then, $\mathcal{M} = \mathcal{M}^{\mathrm{i}}(\mathcal{A})$ is $\langle \mathcal{F}, \mathcal{H} \rangle$ where $\mathcal{F} = \{D_\Gamma : \Gamma \in \mathcal{G}\}$ and $\mathcal{H}$ consists of the homomorphisms just mentioned.

Theorem 9 *Let A be a sentence in $\hat{P}(\Gamma)$. Then*

$$\mathcal{M}, D_\Gamma \models A^i \qquad \textit{iff} \qquad \Gamma \models A.$$

Proof. The proof is immediate if A is atomic. Also, $\wedge, \vee, \exists$ will work in a parallel way. Suppose $A = \neg B$. Then, $A^{\mathrm{i}} = \Box\neg B^{\mathrm{i}}$. We have:

$$\begin{array}{lll} \Gamma \models \neg B & \text{iff} & \text{for all } \Gamma^*, \quad \Gamma^* \not\models B \\ & \text{iff} & \text{for all } D_{\Gamma^*}, \quad \mathcal{M}, D_{\Gamma^*} \not\models B \text{ (ind. hyp)} \\ & \text{iff} & \text{for all } D_{\Gamma^*}, \quad \mathcal{M}, D_{\Gamma^*} \models \neg B \\ & \text{iff} & \mathcal{M}, D_\Gamma \models \Box\neg B^{\mathrm{i}} \end{array}$$

$A = B \to C$ and $A = (\forall x)B(x)$ are similar. ∎

Suppose now that $\mathcal{M} = \langle \mathcal{F}, \mathcal{H} \rangle$ is a D-structure which is a category. We construct a Kripke structure corresponding to $\mathcal{M}$. Given $D \in \mathcal{F}$, a *selection* S for D is a set of maps into D such that if there are any maps $D' \to D$ there is just one such map in S. Take $\mathcal{G} =$ the set of all pairs $< D, S >$, where $D \in \mathcal{F}$ and S in a selection for D. For $\Gamma = \ < D, S > \ \in \mathcal{G}$, take $P(\Gamma) = |D|$ and an atomic sentence A in $\mathcal{P}(\Gamma)$ is forced by Γ iff it holds in D. We let $\Gamma R \Gamma'$ iff there is a map $g \in S'$, $g : D \to D'$ such that for all $f \in S$, $f \circ g \in S'$. (We point out that given $g : D \to D'$ there is always such an S'.)

Theorem 10 *For A in the language of IPC with constants from D, with $\Gamma = \langle D, S \rangle$,*

$$\Gamma \models A \qquad \textit{iff} \qquad \mathcal{M}, D \models A^{\mathrm{i}}.$$

Proof. Quite routine. To check one case, suppose $A = \neg B$. Then, $A^{\mathrm{i}} = \Box\neg B^i$. Then,

$$
\begin{array}{lll}
\Gamma\models A & \text{iff} & \forall\Gamma^*, \quad \Gamma^* \not\models B \\
& \text{iff} & \forall D' \text{ with allowable } g: D \to D', \quad \mathcal{M}, D' \not\models B^i \\
& \text{iff} & \forall D' \text{ with allowable } g: D \to D', \quad \mathcal{M}, D'\models\neg B^i \\
& \text{iff} & \mathcal{M}, D\models\Box\neg B^i \\
& \text{etc.} &
\end{array}
$$

∎

Definition 11 Let $\mathcal{A}$ be a (classical) structure of type μ and f a permutation o $|\mathcal{A}|$. Then $f(\mathcal{A})$ is the structure with base set $|\mathcal{A}|$ in which

$$f(\mathcal{A})\models R(f(a_1), f(a_2), \ldots, f(a_n)) \qquad \text{iff} \qquad \mathcal{A}\models R(a_1, a_2, \ldots, a_n),$$

where $R \in u$ and $a_1, a_2, \ldots, a_n \in |\mathcal{A}|$. A *regular $*$-structure* over $\mathcal{A}$ is a family $\{f(\mathcal{A}) \mid f \in G\}$, where G is some group containing all finite permutations of $|\mathcal{A}|$.

Definition 12 Let $\mathcal{M}$ be a family of first order structures all of the same type μ and with the same base set X. If $X_0 \subseteq X$, $M \in \mathcal{M}$ then

$$\mathcal{M}[X_0, M] = \{N \mid N \in \mathcal{M} \text{ and } N|_{X_0} = M|_{X_0}\}$$

Definition 13 (Ehrenfeucht) Let $\mathcal{M}$ be a regular $*$-structure on $\mathcal{A}$. $X_0 \subseteq |\mathcal{A}|$, $M \in \mathcal{M}$. A is a sentence of $\mathcal{L}_{\mu * X_0}$. We define $\mathcal{M}[X_0, M]\models A$ by induction on $c(A)$.

1. $c(A) = 0$. Then $\mathcal{M}[X_0, M]\models A$ iff $M\models A$.
   (Note: this depends only on $M|_{X_0}$.)
2. $A = B \wedge C$. Then
   $\mathcal{M}[X_0, M]\models A$ iff $\mathcal{M}[X_0, M]\models B$ and $\mathcal{M}[X_0, M]\models C$.
3. $A = B \vee C$. Then
   $\mathcal{M}[X_0, M]\models A$ iff $\mathcal{M}[X_0, M]\models B$ or $\mathcal{M}[X_0, M]\models C$.
4. $A = \neg B$. Then
   $\mathcal{M}[X_0, M]\models A$ iff $\mathcal{M}[X_0, M] \not\models B$.
5. $A = (\exists x)B(x)$. Then
   $\mathcal{M}[X_0, M]\models A$ iff there exist $a \in X$, $b \in X_0 \cup \{a\}$, $N \in \mathcal{M}[X_0, M]$ such that $\mathcal{M}[X_0 \cup \{a\}, N]\models B(b)$.
6. $A = (\forall x)B(x)$. Then
   $\mathcal{M}[X_0, M]\models A$ iff for all $a \in X$, $b \in X_0 \cup \{a\}$, $N \in \mathcal{M}[X_0, M]$, $\mathcal{M}[X_0 \cup \{a\}, N]\models B(b)$.

Theorem 14 *Let $\mathcal{M}$ be a regular $*$-structure on $\mathcal{A}$. Let $\mathcal{M}_1 = \langle \mathcal{F}, \mathcal{H} \rangle$ be defined as follows*

$$
\begin{array}{rl}
\mathcal{F} = & \textit{all finite submodels } D_i \textit{ of } \mathcal{A}, \\
\mathcal{H} = & \textit{all monomorphisms } D \to D' \textit{ with } \overline{\overline{D'}} - \overline{\overline{D}} \leq 1.
\end{array}
$$

Let $X_0 = \{a_1, a_2, \dots, a_n\}$*,* $A(a_1, a_2, \dots, a_n) \in \mathcal{L}_{u*X_0}$*,* $M \in \mathcal{M}$ *and*

$$b_1, b_2, \dots, b_n \in |\mathcal{A}| \quad \textit{such that} \quad \mathcal{A}|_{b_1,b_2,\dots,b_n} = M|_{a_1,a_2,\dots,a_n}.$$

Then

$$\mathcal{M}[X_0, M] \models A(a_1, a_2, \dots, a_n) \quad \textit{iff} \quad \mathcal{M}_1, D \models A^c(b_1, b_2, \dots, b_n),$$

where $\{b_1, b_2, \dots, b_n\} \subseteq |D|$*.*

Proof. Trivial if A is atomic, a negation, conjuction, or disjunction.

Suppose $A = (\forall x)B(x)$. Then, $\mathcal{M}[X_0, M] \models A(a_1, a_2, \dots, a_n)$ gives, for all N, a, b as provided,

$$\mathcal{M}[X_0 \cup \{a\}, N] \models B(a_1, a_2, \dots, a_n, b).$$

Now, let $g : D \to D'$ be an allowable map. We need to show that

$$\mathcal{M}, D' \models B^c(g(b_1), \dots, g(b_n), c), \quad \text{for all } c \in |D'|.$$

Now, there is a permutation ϕ such that $\phi(g(b_i)) = a_i$. Take $a = \phi(b)$, where $b \in D' - g[D]$, if $D' \neq g[D]$ and let $a \in \{a_1, \dots, a_n\}$ otherwise.Let $b = \phi(c)$. Let $N = \phi(\mathcal{A})$. Then

$$\begin{array}{rcl} N|_{\{a_1,\dots,a_n\}} & = & M|_{\{a_1,\dots,a_n\}} \\ & \simeq & \mathcal{A}|_{\{b_1,b_2,\dots,b_n\}} \\ & \simeq & \mathcal{A}|_{\{g(b_1),g(b_2),\dots,g(b_n)\}}. \end{array}$$

and we get

$$\mathcal{M}[X_0 \cup \{a\}, N] \models B(a_1.a_2.\dots, a_n, b)$$

hence

$$\mathcal{M}, D' \models B^c(g(b_1), \dots, g(b_n), c)$$

Thus

$$\mathcal{M}, D' \models (\forall x)B^c(g(b_1), \dots, g(b_n), x)$$

Hence,

$$\mathcal{M}, D \models \Box(\forall x)B^c(g(b_1), \dots, g(b_n), x)$$

which was to be proved.

The backward argument and the $\exists$ case are quite similar. ∎

We now show that a D-structure $\mathcal{M} = \langle \mathcal{F}, \mathcal{H} \rangle$ corresponds to a regular $*$-structure if

1. $\mathcal{M}$ is weakly directed,
2. $D \in \mathcal{F}$ and $D' \subseteq D \to D' \in \mathcal{F}$,
3. the allowable maps are those monomorphisms $D \to D'$ where

$$\overline{\overline{D'}} = \overline{\overline{D}} \leq 1.$$

Theorem 15 *Let $\mathcal{M}$ be a D-structure as above. Choose a maximal subfamily $\mathcal{K} \subseteq \mathcal{H}^t$ such that $\mathcal{K}$ is closed under composition and $\mathcal{K}$ contains at most one map from any D to D'. Let $\mathcal{A}$ be the direct limit of $\mathcal{F}$ under $\mathcal{K}$, and $\mathcal{M}_1$ a regular $*$-structure on $\mathcal{A}$. Suppose $X_0 \subseteq |\mathcal{A}|$, $M \in \mathcal{M}_1$ and $D, a'_1, a'_2, \ldots, a'_n$ are such that $D \in \mathcal{F}$ and $D|_{\{a'_1, a'_2, \ldots, a'_n\}} \simeq M|_{X_0}$. Then,*

$$\mathcal{M}_1[X_0, M] \models A(a_1, a_2, \ldots, , a_n) \quad \textit{iff} \quad \mathcal{M}, D \models A^c(a'_1, a'_2, \ldots, a'_n).$$

Proof. The proof is straightforward. ∎

2 A Game Theoretic Characterisation

Let μ be a relational type, $\mathcal{M}$ a D-structure of type μ, $D \in \mathcal{M}$, $\mathcal{L} = \mathcal{L}^{\mathcal{M}}_{\mu * D}$ the language of modal logic (with quantifiers) and nonlogical symbols from μ and $|D|$, A a closed formula of $\mathcal{L}$. We define a game $\mathcal{G}_{A,D}$ by induction on the complexity of A. (1), (2) are two players.

1. A is atomic. $\mathcal{G}_{A,D}$ is won by (1) iff $D \models A$. Otherwise, it is won by (2).
2. $A = B \wedge C$. Player (2) may choose either game $\mathcal{G}_{B,D}$ or $\mathcal{G}_{C,D}$ which is then played.
3. $A = B \vee C$. Player (1) may choose either game $\mathcal{G}_{B,D}$ or $\mathcal{G}_{C,D}$ which is then played.
4. $A = \neg B$. (1) wins $\mathcal{G}_{A,D}$ iff (s)he loses $\mathcal{G}_{B,D}$.
5. $A = (\forall x)B(x)$. Player (2) chooses an $a \in |D|$. The game $\mathcal{G}_{B(a),D}$ is then played.
6. $A = (\exists x)b(x)$. Player (1) chooses an $a \in |D|$. The game $\mathcal{G}_{B(a),D}$ is then played.
7. $A = \Box B(a_1, a_2, \ldots, a_n)$. Player (2) chooses an $f : D \to D', f \in \mathcal{H}$. The game $\mathcal{G}_{D',B(f(a_1),f(a_2),\ldots,f(a_n))}$ is then played.
8. Like (7) except player (1) chooses the f.

(In 7, the elements of $|D|$ are displayed.)

Theorem 16 *$\mathcal{M}, D \models A$ iff player (1) has a winning strategy for $\mathcal{G}_{A,D}$.*

Corollary 17 *Let $\mathcal{M} = \mathcal{G}_M$ be a regular $*$-structure where M is classical and $\mathcal{G}$ is a group containing all finite permutations of $|M|$. Let A closed such that, $\mathcal{M} \models A$. There exists a finite $X \subseteq |M|$ such that if $N = M|_X$ and $\mathcal{G}_1 =$ all permutations of X, then $\mathcal{G}, M \models A$. (This can be called the "finite model property".)*

Proof. Let $l = c(A)$. There are only finitely many possible diagrams of type μ and size $\leq l$ (upto isomorphism). Choose $X_i \subseteq M$ such that $M|_{X_i}$ is a representative of the ith type occuring inside M. Let $X =$ the union of all the X_i. Let $N = M|_X$.

Let $\mathcal{M}_1$ be the D-structure consisting of all diagrams in N with allowable maps being monomorphisms $D \to D'$ with $\overline{\overline{D'}} - \overline{\overline{D}} \leq 1$.

$\mathcal{M}_2$ is the analogous D-structure for M.

Then, clearly, a closed formula of complexity $\leq l$ holds in $\mathcal{M}_1, D$ iff it holds in $\mathcal{M}_2, D$, where D is the empty diagram. Hence, we get

$$\begin{aligned} \mathcal{G}_M \models A \quad & \text{iff} \quad \mathcal{M}_2 \models A \\ & \text{iff} \quad \mathcal{M}_1 \models A \\ & \text{iff} \quad \mathcal{G}, N \models A \end{aligned}$$

using theorem 14. ∎

Theorem 18 (Skolem-Lowenheim theorem for D-structures) *Let $\mathcal{M} = \langle \mathcal{F}, \mathcal{H} \rangle$ be a D-structure. Then there exist countable $\mathcal{F}_1$, $\mathcal{H}_1$, $\mathcal{F}_1 \subset \mathcal{F}$, $\mathcal{H}_1 \subset \mathcal{H}$ such that for all $D \in \mathcal{F}_1$, $A \in \mathcal{L}^M_{\mu * D}$,*

$$\mathcal{M}_1 = \langle \mathcal{F}_1, \mathcal{H}_1 \rangle \models A \qquad \textit{iff} \qquad \mathcal{M} \models A.$$

Moreover, $\mathcal{M}_1$ is rigid, directed, weakly directed as a category etc. iff $\mathcal{M}_1$ is. Thus $\mathcal{M}_1$ corresponds to an intuitionistic, classical, or regular $*$-structure iff $\mathcal{M}$ does.

Proof. Let

$$X = \mathcal{F} \cup \mathcal{H}^t \cup \bigcup \{ |D| \mid D \in \mathcal{F} \}.$$

We look at the classical structure with base set and relations, constants corresponding to these in μ plus some others. Thus for a relation $R(x_1, \ldots, x_n) \in u$ we have a relation $R'(y, x_1, \ldots, x_n)$ which holds iff y is a digram and $R(x_1, \ldots, x_n)$ holds in y. We also have monadic predicates corresponding to $\mathcal{F}$, $\mathcal{H}$, $\mathcal{H}^t$, $\bigcup \{ |D| \mid D \in \mathcal{F} \}$. In addition we have a function f of two arguments such that

$f(x, y)$	$= x(y)$	whenever $x \in \mathcal{H}^t$ and y in some D, where $x : D \to D'$,
	= something *not* an element	if the conditions are not fulfilled.

Then we have the following. For each formula A of $\mathcal{L}^M_{\mu * D}$, there is a formula A' in the language of M with constants from $|D|$, such that

$$\mathcal{M} \models A \qquad \text{iff} \qquad M \models A'.$$

Moreover, there are formulae of M expressing various properties of $\mathcal{M}$ mentioned. Now take a countable substructure M_1 of M and take the $\mathcal{M}_1$ corresponding. ∎

Special cases of this theorem include: classical structures, intuitionistic structures, regular $*$-structures and rigid D-structures. Note that many properties not explicitly mentioned will be elementary in M (possibly after expanding the language) and will be inherited by M_1.

Game theoretic arguments can be used to give very direct proofs of many results of [EGGdJ] about regular $*$-structures.

3 The logic of D-structures

We recall the three systems M, M', M'' for modal quantificational logic.

M consists of

1. the axioms and rules for the predicate calculus,
2. the axioms
$$A \to \diamond A$$
$$\Box A \leftrightarrow \neg \diamond \neg A$$
$$\diamond(A \vee B) \leftrightarrow \diamond A \vee \diamond B,$$
3. the rules
$$\text{if} \quad \vdash A \leftrightarrow B \qquad \text{then} \quad \diamond A \leftrightarrow \diamond B$$
and
$$\text{if} \quad \vdash A \qquad \text{then} \quad \Box A.$$

Theorem 19 *All theorems of M are valid in all D-structures.*

Proof. It is clear that the axioms are valid and the rules preserve validity. ∎

The system M' is **S4** and is obtained by adding the axiom $\Box A \to \Box\Box A$. The system M'' is **S5** and is obtained by adding, in addition, the axiom $(\diamond\Box A) \to \Box A$.

Definition 20 $\mathcal{M} = \langle \mathcal{F}, \mathcal{H} \rangle$ is *filtered* if for all allowable maps $f : D \to D'$, $g : D \to D''$ there exist D''', h, k such that the diagram

$$\begin{tikzcd}
& D' \arrow[dr, "h"] & \\
D \arrow[ur, "f"] \arrow[dr, "g"'] & & D''' \\
& D'' \arrow[ur, "k"'] &
\end{tikzcd}$$

commutes. $\mathcal{M}$ is *weakly* filtered if $\langle \mathcal{F}, \mathcal{H}^t \rangle$ is filtered.

Theorem 21 *If $\mathcal{M}$ is a category then $\mathcal{M} \models$***S4**.

Proof. Immediate from the definition. ∎

The converse is not true. Suppose we have a situation

$$\begin{tikzcd}
& D_1' \arrow[dr, "g'"] & \\
D \arrow[ur, "f'"] \arrow[dr, "f"'] & & D_2 \\
& D_1 \arrow[ur, "g"'] &
\end{tikzcd}$$

where $g \circ f$ belongs to $\mathcal{H}$ but $g' \circ f'$ does not. However D_1' is a copy of D_1 as far as D is concerned. Then the structure given above will act logically like a category. We do not know if there are any nontrivial examples.

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