Memories and Knowledge Games

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Abstract

A two-part contribution to Johan van Benthem's Liber/CD-ROM Amicorum: some recollections, and some ideas for our common project with Hans van Ditmarsch on Knowledge Games in the form of a proposal for the semantics of regular static knowledge games.

Contents

1 Introduction

Half a century, 50 years, 600 months, 2609 weeks, 18262 days . . . what is in a number? An occasion to celebrate — call it a milestone, and write a book when the person in question deserves it. This all applies to you, Johan, and here comes my part in it: digging up some memories from the past, and presenting some ideas for the (near) future of our Knowledge Games project with Hans van Ditmarsch.

2 Some recollections

We first met in 1975 at the Seminarium Grondslagen at the University of Amsterdam, you as an assistant professor at the Philosophy department with Martin Löb, I as a graduate student in mathematics in Anne Troelstra's group. The topic of the seminar was Nonstandard Models of Arithmetic, and I received my baptism of fire in presenting a survey paper by Dana Scott. I recall your casual remark: why did Scott bother to eliminate some application of the Axiom of Choice, while he also used it implicitly in an ultraproduct construction? That was your first (and not your last) eye-opener for me.

In 1976, we went with Dick de Jongh and Anne Troelstra to the yearly meeting with Justus Diller's group in Münster. Dick and I picked you up at the apartment (it happened to be above that of Simon Carmiggelt) where Lida lived: we felt like two Montague's, taking away Romeo from his beloved Juliet. Your Münster talk on modal logic was my introduction to the subject; in the evening, we were introduced to the inner workings of German student pubs downtown Münster.

A year later I followed your course on Modal Logic, which was closed off with a take-home exam. One of the problems in it could be solved in two ways, with proof theory or with model theory. At that time, I strongly preferred proof theory (under the influence of Löb's beautiful course on the subject), but with your apparent preference for model-theoretic methods in mind, I found it more tactical to take the latter way. So I was somewhat surprised by your remark 'OK, but why not a proof-theoretical argument?'.

In 1977, you went to Groningen, and we saw each other occasionally at meetings of the Vereniging voor Logica. In 1985, I went to Utrecht and you returned to Amsterdam. Some common activities that come to mind are Piet Rodenburg's Ph.D. defence, and the Logic Saturday Afternoon in your garden in Bloemendaal. After 1991, when I went to Groningen, our contacts intensified during the foundation of the Research School in Logic, partly in the shadow of interpersonal and loyalty problems at the Mathematics & Computer Science department in Amsterdam. But these clouds seem to have dissolved for the greater part, and nowadays we see a successful research school (with your Spinoza project shining off on it) leading to many professional and amical contacts, for example in the Knowledge Games project with Hans van Ditmarsch. And that is where the inspiration for the next section comes from.

3 Knowledge games

A game has players $a, b, \ldots \in A$, game situation $s, \ldots \in S$ and moves $\mu, \ldots \in S \to S$. In games with perfect knowledge (tic-tac-toe, draughts, chess, go), all players know the game situation at all times. But in many games (mastermind, almost all card games), this is not the case, and the amount of knowledge that every player has about the game situation becomes part of the extended game situation, that is the factual game situation and the (possibly nested) knowledge about it. Such games are called knowledge games. Most knowledge games are *regular*: every exchange of information between one or more players is globally (i.e., not in full detail) commonly known to the group of all players. In other words: there is no secret information exchange. Here, we have a look at regular knowledge games that are static, i.e. where the factual game situation does not change during the game, only the knowledge about it. Examples are mastermind and cluedo.

In regular static knowledge games, moves have local and global effect: locally, they add common knowledge to the subgroup of players involved in the move; globally, they add common knowledge to the group of all players, namely that some knowledge has been shared by the subgroup. We can encode that by representing move μ by

$$\mu = \langle B, X, \Gamma \rangle \quad (B \subseteq A, X \in \Gamma \subseteq \wp(S))$$

with the effect that, after μ, it is common knowledge for subgroup B that the actual game situation belongs to $X \subseteq S$, and it is common knowledge to all players that those in B share the knowledge represented by some $Y \in \Gamma \subseteq \wp(S)$. In formula:

$$[\mu](C_B\theta_X \wedge C_A(\bigvee\{C_B\theta_Y \mid Y \in \Gamma\}))$$

Here $[\mu]$ is to be read as 'after move μ', C_B is the common knowledge operator 'it is common knowledge among the members of B that ...', and θ_X is the propositional formula characterising $X \subseteq S$, i.e. $X = \{s \in S \mid \theta_X \text{ holds in } s\}$.

We model this with collections of worlds (i.e. extended game situations), where every world encodes information about the game situation and the knowledge that results from the moves that have been played. So we have worlds $w = \langle s, \lambda, \gamma \rangle$ with

$$s \in S,\ \lambda \in \wp^+(A) \to \wp(S),\ \gamma \in \wp^+(A) \to \wp(\wp(S))$$

(using $\wp^+(A) := \wp(A) - \{\emptyset\}$). So λ represents the local knowledge $\theta_{\lambda(B)}$ of every subgroup B, γ the global knowledge $\bigvee\{C_B\theta_Y \mid Y \in \gamma(B)\}$ about the local knowledge of every B. We call world $w = \langle s, \lambda, \gamma \rangle$ consistent iff

$$\forall B \in \wp^+(A)(s \in \lambda(B) \in \gamma(B))$$

i.e. the actual game situation s is possible for every subgroup B, and the actual common knowledge of B is one of the alternatives in the global knowledge. Let W be the collection of consistent worlds. In order to make a Kripke model out of W, we add accessibility relations $R_a, R_\mu \in W^2$ for $a \in A$ and moves $\mu = \langle B, X, \Gamma\rangle$:

$$R_a = \{(\langle s, \lambda, \gamma\rangle, \langle s', \lambda', \gamma'\rangle) \in W^2 \mid \lambda_a = \lambda'_a \;\&\; \gamma = \gamma'\}$$

$$R_\mu = \{(w, w') \in W^2 \mid w = \langle s, \lambda, \gamma\rangle, w' = \langle s, \lambda \sqcap (B \mapsto X), \gamma \sqcap (B \mapsto \Gamma)\rangle\}$$

Here, $\sqcap$ denotes greatest lower bound, so

$$\begin{array}{lcl}
(\lambda \sqcap (B \mapsto X))B & = & \lambda(B) \cap X \\
(\lambda \sqcap (B \mapsto X))C & = & \lambda(C) \quad (C \neq B) \\
(\gamma \sqcap (B \mapsto \Gamma))B & = & \{Y \cap Z \mid Y \in \gamma(B), Z \in \Gamma\} \\
(\gamma \sqcap (B \mapsto \Gamma))C & = & \gamma(C) \quad (C \neq B)
\end{array}$$

It is clear that the R_a are equivalence relations, so we are in multimodal S5. Furthermore, we define

$$\langle s, \lambda, \gamma\rangle \models \theta_X \;:=\; s \in X$$

Now we claim

$$w = \langle s, \lambda, \gamma\rangle \Rightarrow w \models C_B\theta_{\lambda(B)} \;\&\; C_A(\bigvee\{C_B\theta_Y \mid Y \in \gamma(B)\})$$

For the converse of this implication, some work has to be done, viz. formulating an equivalence relation on W that identifies worlds forcing the same formulae. It seems also reasonable to expect

$$\models [\langle B, X, \Gamma\rangle]\varphi \leftrightarrow ((C_B\theta_X \;\&\; C_A(\bigvee\{C_B\theta_Y \mid Y \in \Gamma\}) \rightarrow \varphi)$$

We illustrate the definitions with a simplified version of cluedo. There are six players a, b, c, d, e, f (so $A = \{a, b, c, d, e, f\}$) and 21 cards ($K = \{1, \ldots, 21\}$), and every player receives three cards (represented by an element of $\wp_3(K) := \{X \subseteq K \mid \#X = 3\}$). The three remaining cards are closed on the table, and every player tries to find out the identity of these three closed cards. The game situations are

$$S = \{s : A \rightarrow \wp_3(K) \mid \forall x, y \in A(x \neq y \rightarrow s(x) \cap s(y) = \emptyset)\}$$

and the following moves are possible.

- Initial moves: player x looks into her cards and sees that she has the cards in $M \in \wp_3(K)$ in her hands; all other players see this happen, so there is common knowledge among A that x knows which three cards she has. This can be encoded by

  $$\mu = \langle \{x\}, X^x_M, \{X^x_N \mid N \in \wp_3(K)\}\rangle$$

  with $X^x_M = \{s \in S \mid M \subseteq s(x)\}$, i.e. the set of game situations where x has M.

- Affirmative moves: player x asks player y 'do you have one of the cards in M?', y answers 'yes' and shows x one of the cards m in M. Now x and y have common knowledge that y has card m, and there is common knowledge among A that x and y have common knowledge about y having one of the three cards in M. This is encoded by

$$\mu = \langle \{x, y\}, X^y_{\{m\}}, \{X^y_{\{n\}} \mid n \in M\} \rangle.$$

- Negative moves: player x asks player y 'do you have one of the cards in M?' and y answers 'no'. This is encoded by

$$\mu = \langle A, X^y_{\neg M}, \{X^y_{\neg M}\} \rangle$$

where $X^y_{\neg M} = \{s \in S \mid M \cap s(y) = \emptyset\}$. This is an example of a move with only global knowledge exchange, resulting in common knowledge among A that y has none of the cards in M.

So far this setup for the analysis of regular knowledge games. Questions abound: what is the right equivalence relation on W? what is the relation with Dynamic Epistemic Logic ([3, 4]) and with Baltag's work ([1, 2])? how to deal with dynamic actions, changing the factual game situation? how to incorporate probability? how to formulate and compare strategies? can it be implemented?

References

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