ON STRONG NEIGHBOURHOOD COMPLETENESS OF MODAL AND INTERMEDIATE PROPOSITIONAL LOGICS (Part II)*

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Abstract

The property of strong neighbourhood completeness was introduced in the first part of this paper. By modifying the ultrabouqet construction, we prove this property for all Kripke-complete normal **K4**-logics and for a large class of Kripke-complete polymodal logics (called ‘acyclic). On the other hand, we present a simple counterexample of a polymodal logic with the f.m.p. which is not strongly neighbourhood complete.

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1 Introduction

This paper is a continuation of [8], but a happy occasion on which it has been written – Johan Van Benthem's 50th birthday – suggests us to look at the topic from a more general viewpoint. Indeed, it is closely related to Correspondence Theory, a large area in modal logic discovered 25 years ago, mainly in the works of Johan Van Benthem. The original problems studied by Correspondence Theory were definability of first-order properties and classes of frames by modal formulas , cf. the introduction to [2]. But nowadays we could understand Correspondence Theory in a broader sense, as studying questions of the following two kinds:

- What features of classical first-order formulas (theories) are modal?
- What features of modal formulas (logics) are first-order and what are high-order?

Questions of the first type are considered in the recent book [1]. This paper refers to the second type; it studies a modal logic analogue of compactness.

The strong neighbourhood completeness property we are interested in, was proved in [8] for all Kripke-complete intermediate logics and for many modal logics above **K4**. Here we prove that Kripke-completeness implies strong neighbourhood completeness for all extensions of **K4** and for all 'acyclic' polymodal logics. On the other hand, we construct a simple counterexample for this implication in the polymodal case.

The terminology and notations from part I are retained here with little variations. Now we consider *n-modal formulas* built up from the set of proposition letters $PL = \{p_1, \cdots, p_m, \cdots\}$, and the connectives $\neg$, $\wedge$, $\Box_1, \cdots, \Box_n$; the connectives $\supset$, $\vee$, $\bot$, $\Diamond_i$ are derived[1] The set of all n-modal formulas is denoted by MF_n.

As usual, the minimal n-modal logic $\mathbf{K}_n$ is the smallest set of n-modal formulas, containing all classical tautologies, the axioms:

$$\Box_i(p \supset q) \supset (\Box_i p \supset \Box_i q),$$

and closed under Substitution, Modus Ponens, and $\Box_i$-introduction (for $i \leq n$).

A set $S \subseteq MF_n$ is *consistent* with respect to a modal logic $\mathbf{L}$ (or $\mathbf{L}$-*consistent*) if $\neg(A_1 \wedge \cdots \wedge A_n) \notin \mathbf{L}$ whenever $A_1, \cdots, A_n \in S$.

An *n-modal neighbourhood frame* is a tuple $\mathcal{X} = (X, \Box_1, \cdots, \Box_n)$, where X is a non-empty set, $\Box_i$ are unary operations on $P(X)$ (the power set of X), satsfying the identities:

$$\Box_i(Y \cap Z) = \Box_i Y \cap \Box_i Z,$$

$$\Box_i X = X.$$

A *neighbourhood model* over $\mathcal{X} = (X, \Box_1, \cdots, \Box_n)$ is a pair $(\mathcal{X}, \phi)$, with a mapping

[1] We use the notations $\Box$ and $\Diamond$ (without the subscript 1) in the 1-modal case

$\phi : PL \to P(X)$. ϕ is extended to all modal formulas:

$$\phi(\neg A) = X - \phi(A),\ \phi(A \wedge B) = \phi(A) \cap \phi(B),\ \phi(\Box_i A) = \Box_i \phi(A).$$

A modal formula A is *true* at a point x of a model $(\mathcal{X}, \phi)$ if $x \in \phi(A)$ (notation: $\mathcal{X}, \phi, x \models A$; or just $x \models A$); it is *valid* in $\mathcal{X}$ (notation: $\mathcal{X} \models A$) if it is true at any point of any model over $\mathcal{X}$. A set of formulas S is said to be *valid* in $\mathcal{X}$ (notation: $\mathcal{X} \models S$) if every formula from S is valid. In this case $\mathcal{X}$ is said to be a *(neighbourhood) S-frame* . A set S is *satisfied at a point x of a neighbourhood model* $(\mathcal{X}, \phi)$ if $\mathcal{X}, \phi, x \models A$ for any $A \in S$ (notation: $\mathcal{X}, \phi, x \models S$); S is *satisfied* in $\mathcal{X}$ if it is satisfied in some point of some neighbourhood model over $\mathcal{X}$. It is well-known that every Kripke frame $F = (W, R_1, \cdots, R_n)$ is associated with a neighbourhood frame $\mathcal{X}(F) = (W, \Box_1, \cdots, \Box_n)$, where $\Box_i Y = \{x \mid R_i(x) \subseteq Y\}$; and thus Kripke semantics is reducible to neighbourhood semantics.

Recall that an n-modal logic $\mathbf{L}$ is *neighbourhood complete* (or *N-complete*) if for every n-modal formula $A \notin L$ there exists a neighbourhood L-frame in which A is non-valid. $\mathbf{L}$ is *neighbourhood complete* (or *N-complete*) if the same is true in Kripke semantics.

A class Φ of n-modal Kripke (or neighbourhood) frames *determines* the modal logic $ML(\Phi) = \{A \in MF \mid \forall F \in \Phi\ F \models A\}$. A logic is K-complete iff it is determined by some class of Kripke frames; similarly, for N-completeness.

$\mathbf{L}$ is *strongly neighbourhood* (respectively, *Kripke-*) *complete* (in brief, *S-N-complete; S-K-complete*) if every $\mathbf{L}$-consistent set of n-modal formulas is satisfied in some neighbourhood $\mathbf{L}$-frame. [2]

Note that every logic determined by an elementary class of frames is always S-K-complete, so the difference between these properties can be seen only in the 'non-elementary area'.

In the most general case we have the following diagram:

$$\begin{array}{ccc} \text{S-K-completeness} & \longrightarrow & \text{K-completeness} \\ \downarrow & & \downarrow \\ \text{S-N-completeness} & \longrightarrow & \text{N-completeness} \end{array}$$

It is known that K-completeness does not imply S-K-completeness (cf. [8] for the references). Also it is true that S-N-completeness does not imply K-completeness (even for intermediate logics). This follows from [8] and [7]; the detailed proof will be published elsewhere. In this paper we show that K-completeness does not imply S-N-completeness in the polymodal case. On the other hand, we prove that for the monomodal logics containing $K4$:

$$\begin{array}{cccc} \text{S-K-completeness} & \longrightarrow & \text{K-completeness} & \longrightarrow \\ \text{S-N-completeness} & \longrightarrow & \text{N-completeness} & \\ & (\longleftarrow ?) & & \end{array}$$

The question whether N-completeness implies S-N-completeness for logics above $K4$, remains open. However this implication can be confirmed in many cases ([8], Theorem 5.8).

[2] In [8] the same definitions were given also for intermediate logics. However the Definition 1.3 in [8] contains a misprint. Actually it should read as follows:

an intermediate logic $\mathbf{L}$ is S-N-complete if every $\mathbf{L}$-consistent pair of sets of intuitionistic formulas is satisfied in some neighbourhood $\mathbf{L}$-frame.

2 Ultrabouqets of transitive frames.

The main tools used in [8] in strong completeness proofs were ultrabouqets, the modal logic analogues of ultraproducts. They were defined for transitive antisymmetric (monomodal) Kripke frames ('trantises'). Now we will extend the definition to arbitrary transitive frames.

First, let us recall that, for an ultrafilter $\mathcal{U}$ in ω and a statement $\Phi(n)$ of our meta-language, $\forall^\infty n\ \Phi(n)$ abbreviates $\{n \mid \Phi(n)\} \in \mathcal{U}$. The "generalized quantifier $\forall^\infty$ distributes over all propositional connectives.

Recall also that a *cluster* in a transitive Kripke frame (W, R) is an equivalence class w.r.t. the relation $\{(x, y) \mid xRy \wedge yRx \vee x = y\}$; an *irreflexive cluster* consists of a single irreflexive point.

Consider a countable family of transitive frames $F_n = (X_n, R_n)$, $n \in \omega$. Assume that each F_n contains the least reflexive cluster C_n(and thus, $X_n = R(C_n)$). Take an ultrafilter $\mathcal{U}$ in ω, and let $C = \prod_{\mathcal{U}} C_n$ be the corresponding ultraproduct of sets. For every sequence $(a_n)_{n\in\omega}$, let $[a_n]_{n\in\omega}$ (or, $[a_n]$ in brief) be the corresponding element of C (so $[a_n]_{n\in\omega} = [b_n]_{n\in\omega}$ iff $\forall^\infty n\ a_n = b_n$). Without any loss of generality we may assume that $C \cap X_n = \emptyset$ for every n. Now we glue all F_n together, by identifying every C_n with C.

Speaking precisely, let

$$X_n^- = X_n - C_n, \quad X = C \cup \bigcup_n (\{n\} \times X_n^-).$$

Every $a \in X_n^-$ can be identified with $(n, a) \in X$; so we can consider X_n^- as a subset of X. Let R be the relation in X such that

$$(m, x)R(n, y) \Leftrightarrow m = n \ \&\ xR_n y; \quad R(C) = X.$$

Then obviously, R is transitive.

For any $V \subseteq X$ let

$$\Box V = V_1 \cup V_0,$$

where

$$V_1 = \{x \mid x \notin C \ \&\ R(x) \subseteq V\},$$

$$V_0 = \begin{cases} \{u\}, & \text{if } \forall^\infty n\ X_n^- \subseteq V \text{ and } C \subseteq V; \\ \emptyset & \text{otherwise.} \end{cases}$$

Similarly to [8], Lemma 2.4 one can show that $(X, \Box)$ is a neighbourhood **K4**-frame.

Definition 2.1 *The neighbourhood frame $(X, \Box)$ defined above is called the* ultrabouqet of the family $(F_n)_{n\in\omega}$ *w.r.t. the ultrafilter $\mathcal{U}$ and denoted by* $\bigvee_{\mathcal{U}} F_n$.

Definition 2.2 *In the above context, let ψ_n be a valuation in $F_n = (X_n, R_n)$, ψ be a valuation in $(X, \Box) = \bigvee_{\mathcal{U}} F_n$ such that for any proposition letter p,*

$$\forall n\ \psi(p) \cap X_n^- = \psi_n(p) \cap X_n^-;$$

$$[c_n] \in \psi(p) \Leftrightarrow \forall^\infty n\ c_n \in \psi_n(p).$$

Then ψ is called the ultrabouqet *of the family $(\psi_n)_{n\in\omega}$ and denoted by $\bigvee_{\mathcal{U}} \psi_n$.*

It is easily checked that the family (ψ_n) uniquely defines ψ.

Lemma 2.3 *Let ψ_n , F_n be the same as in the previous Definition, and let $\psi = \bigvee_{\mathcal{U}} \psi_n$. Then for any modal formula A,*
(1) $\forall n\ \psi(A) \cap X_n^- = \psi_n(A) \cap X_n^-$;
(2) $[c_n] \in \psi(A) \Leftrightarrow \forall^\infty n\ c_n \in \psi_n(A)$.

PROOF. By induction, similar to [8], Lemma 2.8. Let us consider the case $A = \Box B$ in (2).

$$[c_n] \in \psi(\Box B) \Leftrightarrow C \subseteq \Box\psi(B) \Leftrightarrow C \subseteq \psi(B)\ \&\ \forall^\infty n\ X_n^- \subseteq \psi(B)$$

$$\Leftrightarrow C \subseteq \psi(B)\ \&\ \forall^\infty n\ X_n^- \subseteq \psi_n(B).$$

Also

$$C \subseteq \psi(B) \Leftrightarrow \forall^\infty n\ C_n \subseteq \psi_n(B).$$

In fact, assume that $C \subseteq \psi(B)$, but not $\forall^\infty n\ C_n \subseteq \psi_n(B)$. Then $\forall^\infty n\ C_n \not\subseteq \psi_n(B)$, and thus there exists a sequence of $d_n \in C_n$ such that $\forall^\infty n\ d_n \notin \psi_n(B)$ (for, take an element in $(C_n - \psi_n(B))$ whenever this set is nonempty; otherwise take an arbitrary element in C_n). By the inductive hypothesis we obtain that $[d_n] \notin \psi(B)$, in a contradiction with $C \subseteq \psi(B)$. So we have

$$[c_n] \in \psi(\Box B) \Leftrightarrow \forall^\infty n\ C_n \subseteq \psi_n(B)\ \&\ \forall^\infty n\ X_n^- \subseteq \psi_n(B) \Leftrightarrow \forall^\infty n\ C_n \subseteq \psi_n(\Box B)$$

$$\Leftrightarrow \forall^\infty n\ c_n \in \psi_n(\Box B). \quad \dashv$$

Lemma 2.4 *Let A be a modal formula, $F_n = (X_n, R_n)$, $n \in \omega$ be* **K4** *-frames validating A. Assume that each F_n contains the least reflexive cluster C_n. Then A is valid in any ultrabouqet $\bigvee_{\mathcal{U}} F_n$.*

PROOF. Suppose the contrary, i.e. that $\varphi(A) \neq X$, for some valuation φ in $(X, \Box) = \bigvee_{\mathcal{U}} F_n$. If $X_n^- - \varphi(A) \neq \emptyset$ for some n, then the generated subframe argument shows that A is non-valid in F_n. Thus $c_1 \notin \varphi(A)$ for some $c_1 \in C$. Let $p_1, \cdots, p_m$ be the list of all proposition letters occurring in A, and consider the following equivalence relation in C:

$$x \equiv_0 y \Leftrightarrow \forall i \leq m\ (x \in \varphi(p_i) \Leftrightarrow y \in \varphi(p_i)).$$

First let us assume that

$$(X, \Box), \varphi, c_1 \models \neg p_1 \wedge \cdots \wedge \neg p_m. \tag{1}$$

Let $c_1, \cdots, c_k$ be the maximal set of non-equivalent points in C (w.r.t $\equiv_0$). Consider the valuation ψ in X, such that for every $q \in PL$

$$x \in \psi(q) \Leftrightarrow x \in \varphi(q),\ \text{if } x \in (X - C) \cup \{c_1, \cdots, c_k\};$$

$$x \notin \psi(q),\ \text{if } x \in C - \{c_1, \cdots, c_k\}.$$

For every c_i choose a sequence $(c_{in})_{n\in\omega}$, such that $c_i = [c_{in}]_{n\in\omega}$, and take the valuations ψ_n in F_n such that for every q

$$\psi_n(q) \cap C_n = \{c_{in} \mid c_i \in \varphi(q)\}, \quad \psi_n(q) \cap X_n^- = \psi(q) \cap X_n^- .$$

Then we have:

$$[a_n] \in \psi(q) \Leftrightarrow \forall^\infty n \; a_n \in \psi_n(q). \tag{2}$$

In fact, by the definition of ψ, ψ_n and the properties of $\forall^\infty$,

$$\forall^\infty n \; a_n \in \psi_n(q) \Leftrightarrow \forall^\infty n \; (a_n = c_{1n} \;\&\; c_1 \in \psi(q) \vee \cdots \vee a_n = c_{kn} \;\&\; c_k \in \psi(q))$$
$$\Leftrightarrow \forall^\infty n \; (a_n = c_{1n} \;\&\; c_1 \in \psi(q)) \vee \cdots \vee \forall^\infty n \; (a_n = c_{kn} \;\&\; c_k \in \psi(q))$$
$$\Leftrightarrow \exists i \leq k([a_n] = c_i \;\&\; c_i \in \psi(q)) \Leftrightarrow [a_n] \in \psi(q).$$

$$c_1 \notin \psi(A) \Rightarrow \forall^\infty n \; F_n \not\models A. \tag{3}$$

For, $\psi = \bigvee_{\mathcal{U}} \psi_n$ by (1), and thus $c_1 \notin \psi(A)$ implies $\forall^\infty n \; c_{1n} \notin \psi_n(A)$ by Lemma 2.3.

$$c_1 \notin \psi(A). \tag{4}$$

To verify this, we show by induction that for any formula E in $p_1, \cdots, p_m$

$$\forall i \leq k \; (c_i \in \psi(E) \Leftrightarrow c_i \in \varphi(E)). \tag{5}$$

Only the case $E = \Box B$ is non-trivial. We have

$$c_i \in \psi(\Box B) \Leftrightarrow C \subseteq \psi(B) \;\&\; \forall^\infty n \; X_n^- \subseteq \psi(B) \Leftrightarrow C \subseteq \psi(B) \;\&\; \forall^\infty n \; X_n^- \subseteq \varphi(B),$$

due to the definition of φ . Now it suffices to prove

$$C \subseteq \psi(B) \Leftrightarrow C \subseteq \varphi(B). \tag{6}$$

($\Rightarrow$) Let $C \subseteq \psi(B)$. Then $c_1, \cdots, c_k \in \varphi(B)$ by (5), and also $d \in \varphi(B)$ for every $d \in C - \{c_1, \cdots, c_k\}$. For, by the choice of $c_1, \cdots, c_k$, we have $d \equiv_0 c_i$ for some i, and an easy inductive argument shows that (w.r.t. the valuation φ) the same formulas in $p_1, \cdots, p_m$ are true in d, c_i .

($\Leftarrow$) Let $C \subseteq \varphi(B)$. Then $c_1, \cdots, c_k \in \psi(B)$ by (5), and again, $d \in \psi(B)$ for every $d \in C - \{c_1, \cdots, c_k\}$. In fact, $d \notin \psi(p_j)$ for $j \leq m$, and thus (w.r.t the valuation ψ and formulas in $p_1, \cdots, p_m$), d is equivalent to c_1 .

This completes the proof of (6) and (5). Now $c_1 \notin \varphi(A)$ implies (4), and from (4) , (3) we get $\forall^\infty n \; F_n \not\models A$, which contradicts the assumption of the Lemma. Therefore (1) is impossible.

Now instead of (1), let us assume that

$$(X, \Box), \varphi, c_1 \models \neg p_1' \wedge \cdots \wedge \neg p_m',$$

where each p_j' is either p_j or $\neg p_j$. Take the valuation φ', such that for every $j \leq m$,
$\varphi'(p_j) = \varphi(p_j')$; then obviously, $\varphi'(p_j') = \varphi(p_j)$; and hence

$$\varphi(A(p_1, \cdots, p_m)) = \varphi'(A(p_1', \cdots, p_m')).$$

Now we have that $A' = A(p_1', \cdots, p_m')$ is valid in every F_n , and also

$$c_1 \notin \varphi'(A'); \quad (X, \Box), \varphi', c_1 \models \neg p_1 \wedge \cdots \wedge \neg p_m,$$

and so the previous argument leads us to a contradiction again. Therefore $\varphi(A) = X$. $\dashv$

3 From K-completeness to S-N-completeness

Theorem 3.1
Every K-complete monomodal logic containing **K4** *is S-N-complete.*

PROOF. Similar to [8], Theorem 3.3. Suppose $\mathbf{L} = \mathbf{ML}(\Phi)$ for a class Φ of transitive 1-modal Kripke frames. Consider an **L**-consistent countable set of formulas $S = \{A_n \mid n \in \omega\}$, then every formula $B_n = \bigwedge_{i=0}^{n} A_i$ is **L**-consistent, and thus there exists a frame $F_n \in \Phi$ and a valuation ϕ_n , such that $F_n, \phi_n, x_n \models B_n$. Obviously, $(B_n \supset B_m) \in \mathbf{K4}$ if $m \leq n$. Now there are two cases.

(i) The set $\{n \mid x_n$ is reflexive $\}$ is infinite.

Let $\{n_1, n_1, \cdots, \}$ be the increasing enumeration of this set; then $n_k \geq k$. Also let $y_k = x_{n_k}$, and let (X_k, ψ_k) be the submodel of (F_{n_k}, ϕ_{n_k}) generated by y_k. By the properties of generated subframes, we have

$X_k \models \mathbf{L}$ and $X_k, \psi_k, y_k \models B_{n_k}$; hence $X_k, \psi_k, y_k \models B_k$(because $n_k \geq k$).

Take some non-principal ultrafilter $\mathcal{U}$ in ω, and consider the ultrabouqet $X = \bigvee_{\mathcal{U}} X_n$. Then $X \models \mathbf{L}$, by Lemma 2.4.

On the other hand, take the valuation $\psi = \bigvee_{\mathcal{U}} \psi_n$. Since we have $X_n, \psi_n, y_n \models C_n$, it follows that (for any k)

$$\forall n \geq k \; y_n \in \psi_n(A_k),$$

and thus

$$\forall^{\infty} n \; y_n \in \psi_n(A_k),$$

because $\mathcal{U}$ is non-principal. Then by Lemma 2.3, $[y_n] \in \psi(A_k)$, and therefore $X, \psi, [y_n] \models S$.

(ii) The set $\{n \mid x_n$ is reflexive$\}$ is finite. Then the set $\{n \mid x_n$ is irreflexive$\}$ is infinite, and we may use the same argument as in the proof of Theorem 3.3 from [8]. $\dashv$

Definition 3.2 *A* cycle *of length $m > 1$ in a Kripke frame $F = (W, R_1, \cdots, R_l)$ is a sequence of distinct points $x_1, \cdots, x_m$ such that $x_1 R x_2 R \cdots R x_{m-1} R x_m R x_1$, where $R = R_1 \cup \cdots \cup R_l$. A Kripke frame without cycles is called* acyclic .

Definition 3.3 *A* reflexifity type *of a point x in a Kripke frame $F = (W, R_1, \cdots, R_l)$ is the set $rt(x) = \{i \mid x R_i x\}$.*

Definition 3.4 *For every $n \in \omega$, let $F_n = (X_n, R_{n1}, \cdots, R_{nl})$ be an acyclic generated frame with a root x_n, and assume that all the x_n are of the same reflexivity type. The* bouqet *of the frames F_n is the frame $(X, R_1, \cdots, R_l)$, where*

$$X = \{u\} \cup \bigcup_n (\{n\} \times X_n^-), \; X_n^- = X_n - \{x_n\}, \; u \notin \bigcup_n X_n,$$

$$(m, x) R_i (n, y) \Leftrightarrow m = n \;\&\; x R_{ni} y; \; u R_i (n, y) \Leftrightarrow x_n R_{ni} y; \; rt(u) = rt(x_1).$$

As usual, X_n^- is identified with $(\{n\} \times X_n^-)$.

For an ultrafilter $\mathcal{U}$ *in* ω, *we define the* ultrabouqet $\bigvee_{\mathcal{U}} F_n$ *as the neighbourhood frame* $(X, \Box_1, \cdots, \Box_l)$, *where*

$$\Box_j V = V_j^0 \cup V_j^1,$$

$$V_j^1 = \{x \mid x \neq u \;\&\; R_j(x) \subseteq V\},$$

$$V_j^0 = \begin{cases} \{u\}, & \textit{if } \forall^\infty n\ R_{nj}(x_n) - \{x_n\} \subseteq V(\textit{ and also } u \in V \textit{ if } uR_j u) \\ \emptyset & \textit{otherwise.} \end{cases}$$

Definition 3.5 *Let* F_n *be the same as in the previous Definition, and for every* n, *let* ψ_n *be a valuation in* F_n. *We define the valuation* $\psi = \bigvee_{\mathcal{U}} \psi_n$ *in* $\bigvee_{\mathcal{U}} F_n$ *such that for any proposition letter* p,

$$\forall n\ \psi(p) \cap X_n^- = \psi_n(p) \cap X_n^-;$$

$$u \in \psi(p) \Leftrightarrow \forall^\infty n\ x_n \in \psi_n(p).$$

The following two lemmas are quite similar to the case considered in [8]:

Lemma 3.6 *Let* $\psi = \bigvee_{\mathcal{U}} \psi_n$. *Then for any* l*-modal formula* A,
(1) $\forall n\ \psi(A) \cap X_n^- = \psi_n(A) \cap X_n^-$ *;*
(2) $u \in \psi(A) \Leftrightarrow \forall^\infty n\ x_n \in \psi_n(A)$.

Lemma 3.7 *In the assumptions of Definition 3.5, let* A *be an* l*-modal formula which is valid in every* F_n. *Then* A *is valid in any ultrabouqet* $\bigvee_{\mathcal{U}} F_n$.

Theorem 3.8 *Every modal logic determined by a class of acyclic Kripke frames is S-N-complete.*

PROOF. Similar to Theorem 3.3 from [8] and to Theorem 3.1 above. Suppose $\mathbf{L} = \mathbf{ML}(\Phi)$ for a class Φ of acyclic Kripke frames. Consider an $\mathbf{L}$-consistent countable set of formulas $S = \{A_n \mid n \in \omega\}$. For $B_n = \bigwedge_{i=0}^n A_i$ there exists a frame $F_n \in \Phi$ and a valuation ϕ_n , such that $F_n, \phi_n, x_n \models B_n$. Then some reflexivity type is represented by infinitely many of x_n. Let $n_1, n_2, \cdots$ be the increasing enumeration of these numbers n, and let (X_k, ψ_k) be the submodel of (F_{n_k}, ϕ_{n_k}) generated by $y_k = x_{n_k}$. For a non-principal ultrafilter $\mathcal{U}$ in ω, consider the ultrabouqet $X = \bigvee_{\mathcal{U}} X_n$ and the valuation $\psi = \bigvee_{\mathcal{U}} \psi_n$. Then $X \models \mathbf{L}$, by Lemma 3.7, and $X, \psi, u \models S$ by Lemma 3.6. $\dashv$

Corollary 3.9 *Every uniform logic (in the sense of [5]) containing* $\mathbf{D}$ *is S-N-complete.*

PROOF. The normal form construction from [5] (cf. also [3]) shows that every uniform logic above $\mathbf{D}$ is determined by a class of finite acyclic frames. $\dashv$

In particular, the famous logic $\mathbf{M} = \mathbf{K} + \Box\Diamond p \supset \Diamond\Box p$ which is "very second-order" in Kripke semantics [2], becomes better in neighbourhood semantics.

4 S-N-incompleteness

Now let us construct a rather natural bimodal logic which is K-complete, but S-N-incomplete. Consider the formula $AGrz = \neg(p \wedge \Box(p \supset \Diamond(\neg p \wedge \Diamond p))$. Then $\mathbf{Grz} = \mathbf{S4} + AGrz$ is the well-known *Grzegorczyk's logic* , determined by the class of all finite partial orders [3].

Definition 4.1 *Let* **GrzU** *be* Grzegorczyk's logic with the universal modality. *It is defined as the logic with modal operators* $\Box$, $\forall$, *and the following postulates:*
(1) postulates of **Grz** *for* $\Box$*;*
(2) postulates of **S5** *for* $\forall$ *;*
(3) $\forall p \supset \Box p$*.*

Theorem 4.2 **GrzU** *has the f.m.p (and therefore is K-complete).*

PROOF. Take a generated submodel $M = (W, R_1, R_2, \varphi)$ of the canonical model refuting a given formula A. Due to the axioms (2) and (3), the relation R_2 is universal. Then we extract a finite submodel of M refuting A by a standrad filtration argument in the same way as it is done for **Grz**. $\dashv$

To show that **GrzU** is S-N-incomplete we use the well-known K. Fines frame FF from [4]. The whole argument is close to Sec. 2 of [7], so we do not give it in full detail here.

Definition 4.3 *By induction we define the formulas* β_n, γ_n *:*

$$\beta_0 = \Box p,\ \gamma_0 = \Box\neg p, \beta_1 = \neg p \wedge \Diamond\beta_0 \wedge \neg\Diamond\gamma_0,\ \gamma_1 = p \wedge \neg\Diamond\beta_0 \wedge \Diamond\gamma_0,$$

$$\beta_{n+1} = \Diamond\beta_n \wedge \Diamond\gamma_{n-1} \wedge \neg\Diamond\gamma_n, \beta_{n+1} = \Diamond\gamma_n \wedge \Diamond\beta_{n-1} \wedge \neg\Diamond\beta_n \ (\textit{for } n \geq 1).$$

Let $\alpha_n = \Diamond\beta_{n+1} \wedge \Diamond\gamma_{n+1} \wedge \neg\Diamond\gamma_{n+2} \wedge \neg\Diamond\beta_{n+2}$, $\epsilon_n = \Diamond\alpha_n \wedge \Diamond\beta_{n+2}$,

$$lpr\theta_n = \epsilon_{n+1} \wedge \neg\Diamond\alpha_n,\ \delta_n = \epsilon_n \supset \Diamond\theta_n.$$

Let $S = \{\epsilon_0\} \cup \{\forall\delta_n \mid n \in \omega\}$.

Definition 4.4 *Let* $FF = (W, \leq)$*, where* $W = \bigcup_n \{a_n, b_n, c_n, d_n\}$*,* $\leq$ *is the least partial order in* W*, such that*

$$b_{n+1} \leq b_n, c_{n-1};\ c_{n+1} \leq c_n, b_{n-1};\ a_n \leq b_{n+1}, c_{n+1};\ d_n \leq a_n, d_{n+1}.$$

Let FF_n *be the restriction of* FF *to the set* $V_n = W - \{d_m \mid m > n\}$.

Lemma 4.5 *The set* S *is* **GrzU***-consistent.*

PROOF. We will show that every set $S_n = \{\epsilon_0\} \cup \{\forall\delta_m \mid m \leq n\}$ is **GrzU**-consistent. Since every ascending chain in FF_n is finite, the logic **GrzU** is valid in FF_n with $\forall$ interpreted as the universal truth. So it is sufficient to show that S_n is satisfied in some Kripke model over FF_n. Take the valuation φ in V_n, such that $\varphi(p) = \{b_0, c_1\}$. Then by induction on k it follows that $\varphi(\beta_k) =$

$\{b_k\},\ \varphi(\gamma_k) = \{c_k\}$. This implies $\varphi(\alpha_k) = \{a_k\}$, and thus $d_0 \in \varphi(\epsilon_0)$. Also $\varphi(\delta_m) = V_n$. For, $x \in \varphi(\epsilon_m)$ implies $x \leq a_m,\ x \leq b_{m+2}$, and thus $x \leq d_m$. But $d_{m+1} \leq a_{m+1}, b_{m+3}, c_{m+3};\ d_{m+1} \not\leq a_m$; therefore $d_{m+1} \in \varphi(\theta_m),\ x \in \varphi(\Diamond\theta_m)$. So we obtain that the formulas $\forall\delta_m$ are true in (FF_n, φ) whenever $m \leq n$, and thus $(FF_n, \varphi), d_0 \models S_n$. ⊣

Lemma 4.6 *S is not satisfied in any neghbourhood frame validating* **GrzU**.

PROOF. Suppose the contrary, and let $\mathcal{X}$ be a neighbourhood **GrzU**-frame satisfying S. Due to the postulates (2), (3) of **GrzU**, $\mathcal{X}$ is isomorphic to a disjoint union of topological spaces $\mathcal{X}_i$, such that $\Box$ is interpreted in $\mathcal{X}_i$ as the interior operator, and $\forall$ as the universal truth. So without any loss of generality, we may assume that $\mathcal{X}$ is one of those $\mathcal{X}_i$.

Then $\varphi(\epsilon_0) \neq \emptyset,\ \varphi(\forall\delta_n) = X$, for some valuation φ in $\mathcal{X}$. Let $X_0 = \varphi(\epsilon_0),\ X_n = \varphi(\theta_n)$ for $n > 0$. Then $X_n \subseteq \varphi(\epsilon_{n+1}) \subseteq \Diamond W_{n+1}$ (since by our assumption $\varphi(\delta_{n+1}) = X$). Also $X_n \cap X_{n+1} = \emptyset$, since $X_n \subseteq \varphi(\epsilon_{n+1}) \subseteq \varphi(\Diamond\alpha_{n+1}),\ X_{n+1} \subseteq \varphi(\epsilon_{n+1}) - \varphi(\Diamond\alpha_{n+1})$.

Now it follows that $AGrz$ is refuted in $\mathcal{X}$. In fact, let $\mathcal{Y}$ be a subspace of $\mathcal{X}$ on the set $Y = \bigcup_{n\in\omega} X_n$. For $y \in Y$, let $\mu(y) = min\{n \mid y \in X_n\}$.

Let $\eta(y)$ be $\mu(y)$ modulo 2. Then η is an interior mappping from $\mathcal{Y}$ onto the set $\{0, 1\}$ with the weakest topology (i.e. onto the neighbourhood frame of a two-element cluster). To prove this, it is sufficient to show that the subsets $Y_0 = \eta^{-1}(0),\ Y_1 = \eta^{-1}(1)$ are dense in $\mathcal{Y}$. For the latter, we check by induction on n, that

$$\forall n \forall x \in Y(\mu(x) = n \Rightarrow x \in \Diamond Y_0 \cap \Diamond Y_1).$$

In fact, assume this for every $k < n$, and let $\mu(x) = n$; then $x \in X_n \subseteq \Diamond X_{n+1}$. Let V be any neighbourhood of x; take a point $y \in X_{n+1} \cap V$. Assume that n is even. Then $x \in Y_0,\ V \cap Y_0 \neq \emptyset$. Since $X_n \cap X_{n+1} = \emptyset$, we have either $\mu(y) = n+1$ (and thus, $y \in Y_1 \cap V$), or $\mu(y) = n+1$ (and thus $y \in \Diamond Y_1$ by the inductive hypothesis, and again $Y_1 \cap V \neq \emptyset$). The case when n is odd is quite similar. Since η is interior and the formula $AGrz$ is refuted in the two-element cluster, it is refuted in $\mathcal{Y}$. Therefore it is refuted also in $\mathcal{X}$, because the validity of $AGrz$ is preserved by any (not necessarily open) subspace. ⊣

Theorem 4.7 *The logic* **GrzU** *is S-N-incomplete.*

PROOF. From 4.4. and 4.5. ⊣

5 Conclusion

The results obtained in this paper show the crucial difference between the neighbourhood and Kripke semantics. In fact, it turns out that in neighbourhood semantics, modal logics enjoy the *compactness property* in numerous cases, and thus they acquire features of *classical first order theories*. On the other hand, in Kripke semantics modal logics are known to behave like *second order theories*, because such properties as compactness, Löwenheim – Skolem etc. fail too often.

Another important point is the difference between the transitive and the general case. Although the counterexample in Section 4 is bimodal, it is very likely to be reformulated as monomodal, by using Thomason's translations [9], [10], [6]. But it in the transitive case such counterexamples are impossible, due to Theorem 3.1. This leads us again to an old question:

Does there exist a Thomason-style translation from polymodal logics (and consequently, from classical second-order logic[11]) to monomodal **K4***-logics?*

A common opinion among modal logicians has been that this question has an answer 'yes'. However the previous observations seem to point at the answer 'no'.

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