The role of intuitionistic reasoning in the development of the proof mining methodology

dedicated to the memory of Anne S. Troelstra (1939-2019)

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Brief background on Proof Mining

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- Interesting proofs that use WKL (Troelstra, JSL 1974) but allow for a **WKL-elimination**: **uniqueness statements** ($\in \forall \to \forall$).
- Carrying all this out in an extended case study: moduli and constants of **strong unicity in best Chebycheff approximation** (later with P. Oliva also for L^1**-approximation**).

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$$\forall x, y \in C\ (\|Tx - Ty\| \leq \|x - y\|).$$

Consider appropriate iterations such as the Krasnoselski-Mann iteration

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n Tx_n,\ \ x_0 := x \in C,$$

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Under much more general conditions (e.g. (x_n) being bounded), one has **asymptotic regularity**

$$\|x_n - Tx_n\| \to 0.$$

Why rewarding for proof mining?

- $\|x_n - Tx_n\| \to 0$ has form $\forall\exists$ since $(\|x_n - Tx_n\|)$ is decreasing.
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- Numerous similar results not only in fixed point theory but also **ergodic theory, convex optimization, nonlinear semigroup theory** etc.
- The **finitary proof-theoretic analysis** makes it easy to **generalize things to geodesic settings** (with L. Leuştean).
- Extracted bounds are highly uniform: **new qualitative information!**

Since 2004: rates of metastability

If $(\|x_n - Tx_n\|)$ is not monotone or one studies the convergence of (x_n) itself, **in general no computable rate of convergence possible**.

Let (x_n) be a Cauchy sequence in a metric space (X, ρ), i.e.

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \geq n\, (\rho(x_i, x_j) \leq 2^{-k}) \in \forall\exists\forall$$

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A bound $\Phi(k,g)$ on '$\exists n$' in the latter formula is a **rate of metastability** (introduced by **Kreisel** in 1951 as **no-counterexample interpretation).**

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- Huge gap between **ideal principles** used and the **concreteness** of the theorem proven: Cauchy-property (Π^0_3).
- Concrete bounds numerically interesting. Often information on the **algorithmic learnability** of a rate of convergence which - if a gap condition is satisfied - yields **oscillation bounds** (Safarik/K., Avigad/Rute).

Inspiration from Troelstra's work on Intuitionism I:

Closure under Fan Rules (Troelstra JSL 1974,1977)

The fan rule

In his 74/77 JSL-papers, Troelstra proved among many other things:

Theorem (Troelstra 74,77)

Let H^ω be intuitionistic arithmetic in all types or analysis E-HA$^\omega$, N-HA$^\omega$ or EL. Then H^ω is closed under the **fan rule**, i.e.

$$\mathbf{H}^{\omega} \vdash \forall f^1 \exists n^0\, A(f,n) \;\Rightarrow \mathbf{H}^{\omega} \vdash \forall g^1\, \exists n^*\, \forall f \leq g\, \exists n \leq n^*\, A(f,n).$$

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- Troelstra 1974 uses modified realizability (with truth) and the uniform continuity of closed terms $t^2 \in T$.
- Troelstra 1977 uses a notion of **fan computability** and Troelstra/van Dalen 1988 **uniform forcing**.
- K. 1992 uses modified realizability (with truth) and **majorizability** (adopted in Troelstra's 1998 Handbook article).

Theories with abstract metric structures X

Consider extensional Heyting arithmetic $\text{E-HA}^{\omega,X}[X,d,b]$ over all finite types over the base types $\mathbb{N}, X$ where X represents an abstract b-**bounded** metric space with

$$x =_X y := d_X(x,y) =_{\mathbb{R}} 0.$$

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Let

$$\mathbf{CA}_\neg : \exists\Phi\,\forall x(\Phi(x) =_{\mathbb{N}} 0 \leftrightarrow \neg A(x)),$$

where x is an arbitrary tuple of variables of **arbitrary types** and A an **arbitrary formula**.

Let **AC** be the axiom-of-choice schema in all types.

A fan-type rule for abstract spaces

A simple version reads as follows:

Theorem (Gerhardy/K. APAL 2006)

Let ρ (resp. τ) be an arbitrary type with values in $\mathbb{N}$ (resp. in X). s is a closed term. If (for arbitrary A, B)

$$\textbf{E-HA}^{\omega,X}[X,d,b]+\textbf{AC}+\textbf{CA}_{\neg} \vdash \forall x^1 \forall y \leq_\rho s(x) \forall z^\tau (\neg B \to \exists n^{\mathbb{N}} A)$$

then one can extract a functional $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ in Gödel's T s.t.

$$\forall x^1 \forall y \leq_\rho s(x) \forall z^\tau \exists n \leq \Phi(x,b) (\neg B \to A)$$

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Methods: monotone versions of extentions of mr resp. Dialectica.

The usual fan rule as a special case

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- In 1995, I showed that the same is true if noneffective principles such as CA$\neg$ or M$^\omega$+KL (not both!) are added.
- Despite some efforts, I never managed to find an application for $\rho > 1$ in mainstream mathematics. However: there are **many applications for $\tau = X$ and for $\tau = X^X$!**

The fan rule from a classical point of view

Classically (Σ^0_1-LEM), the fan rule seems to fail miserably: consider

$$\forall f \in 2^{\mathbb{N}}\, \exists n \in \mathbb{N}\, \underbrace{\forall k \in \mathbb{N}(f(k) = 0 \to f(n) = 0)}_{\Pi^0_1}.$$

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However: it holds (even in the much generalized form) if $A_\exists$ is **purely existential** (and one weakens the extensionality axiom to a quantifier-free rule and restricts AC to quantifier-free formulas):

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Method: extended version of monotone Dialectica with negative translation.

Other admissible structures X

- Hyperbolic, CAT(0), CAT($\kappa > 0$), normed, their completions, Hilbert, uniformly convex, uniformly smooth (not: separable, strictly convex or smooth) spaces.
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- Also **several** spaces $X_1, \ldots, X_n$ (Günzel/K.)
- All normed structures definable in **positive bounded logic**, e.g. **abstract** L_p and $C(K)$-spaces (Günzel/K. 2016).

The unbounded case

Here z^τ needs to **majorizable** (extending Howard's notion: Gerhardy/K.2008): y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X

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$$f^* \gtrsim^a_{X \to X} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq d_X(a, x) \to f^*(n) \geq d_X(a, f(x))].$$

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Then for $d_X(a, f(a)) \leq b$ and $f^*(n) := n + b$: $f^* \gtrsim^a_{X \to X} f$.
The bound then depends also on f^*.

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Then for $d_X(a, f(a)) \leq b$ and $f^*(n) := n + b$: $f^* \gtrsim^a_{X \to X} f$.

The bound then depends also on f^*.

In a **normed setting:** $a := 0_X$.

Inspiration from Troelstra's work on Intuitionism II:

Conservation results for the Fan Principle (Troelstra JSL 1974)

Let EL^+ be elementary intuitionistic analysis plus $\mathrm{AC}^{\mathbb{N},\mathbb{N}^\mathbb{N}}$ plus a continuity principle CONT_1. Consider FAN in the form (equivalent to Troelstra's definition of FAN over EL^+)

$$\forall f \in 2^{\mathbb{N}}\, \exists n \in \mathbb{N}\, A(f,n) \rightarrow \exists n^* \in \mathbb{N}\, \forall f \in 2^{\mathbb{N}}\, \exists n \leq n^*\, A(f,n).$$

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In Troelstra's 1977 Handbook article: elimination of choice sequences used to show conservativity of FAN.

Conservation of generalized fan principles

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- Even more general forms for uniform boundedness principles are studied in the context of **bounded functional interpretation** as **'bounded collection principles'** (Engracia 2009, Ferreira 2009).

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Recently shown to allow to **replace** some **uses of weak sequential compactness** (Ferreira, Leuştean, Pinto 2019).

Inspiration from Troelstra's work on Intuitionism III:

Troelstra on Extensionality, Intensionality and Continuity (1973-2000)

- Troelstra 1973 for the first time clearly formulated the various options to interpret $=_\rho$ in higher type intuitionistic arithmetic.

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- Troelstra stressed in many papers the role of **enrichment of data** making objects extensional in the enriched data.
- Intensional aspects of choice sequences.

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Important in fixed point theory: mappings with Suzuki's condition

$$\forall x, p \in C(\|p - Tx\| \le \mu\|p - Tp\| + \|x - y\|) \quad (\mu \ge 1).$$

Then $\delta_T(\varepsilon) = \varepsilon/4, \omega_T(\varepsilon) = \varepsilon/(2\mu)$. **No continuity requirement!**

Inspiration from Troelstra's work on Intuitionism IV:

Restricted forms of LEM (e.g. Troelstra/van Dalen 1988)

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- Same for $\mathbf{HA^\omega[X, d, \ldots]+AC+KL+M^\omega}$, where M^ω is Markov's principle in all types and KL is König's lemma (K.1995,2008).
- Naive formalization of the monotone convergence principle uses $\mathbf{\Sigma^0_2}$**-DNE:** $\neg\neg\exists x^{\mathbb{N}}\forall y^{\mathbb{N}}\varphi_{qf}(x, y) \rightarrow \exists x^{\mathbb{N}}\forall y^{\mathbb{N}}\varphi_{qf}(x, y)$, but using more induction, weaker Σ^0_1-LEM$(t(x_n))$ suffices (Toftdal 2004).

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- **Question:** does forbidding nested repeated use of Σ^0_1-LEM suffice to guarantee effective fluctuation bounds? **Later: No! But...**

Effective (B, L)-learnability

Definition (Safarik/K.,2014)

Consider a Σ^0_2 formula $\varphi \equiv \exists n^{\mathbb{N}} \forall x^{\mathbb{N}}\, \varphi_{qf}(x, n, \underline{a})$ which is monotone in n, i.e.

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φ is *(B,L)-learnable*, if there are function(al)s B and L s.t.

$$\exists i \leq B(\underline{a})\ \forall x\ \varphi_{qf}(x, c_i, \underline{a}), \text{ where}$$

$$c_0 := 0,$$

$$c_{i+1} := \begin{cases} L(x, \underline{a}), & \text{for the } x \text{ with } \neg\varphi_{qf}(x, c_i, \underline{a}) \wedge \forall y < x\ \varphi_{qf}(y, c_i, \underline{a}) \text{ if } \exists \\ c_i, & \text{otherwise.} \end{cases}$$

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The **hierarchy is strict** in the sense that the existence of computable witnesses for level n not even follows from primitive recursive witnesses for level $n - 1$ $(2 \le n \le 4)$.

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Separation of levels 1 and 2: Specker sequences!

The other separations are more complicated (especially 3 versus 4).

A metatheorem for (B, L)-bounds

Theorem (Safarik/K., 2014)

Let ψ_{qf}, φ_{qf} be quantifier-free s.t. $\varphi := \exists n \forall x\, \varphi_{qf}(x, n)$ is monotone. Suppose $\mathcal{T}$ **:=HA$^\omega$[$X, d, \ldots$]+AC+M$^\omega$+IP$^\omega_{\forall}$.** proves a sentence

$$\forall \underline{a}\, \exists k^{\mathbb{N}} \begin{cases} \Big(\forall m \leq k\ \big(\exists u^{\mathbb{N}}\, \psi_{qf}(u, m, \underline{a}) \vee \forall v^{\mathbb{N}} \neg \psi_{qf}(v, m, \underline{a})\big) \\ \quad \rightarrow \exists n^{\mathbb{N}} \forall x^{\mathbb{N}} \varphi_{qf}(x, n, \underline{a})\Big). \end{cases}$$

Then one can extract by monotone functional interpretation (self-majorizing) primitive recursive (Gödel) functionals B^*, L^* s.t. **φ is (B^*, L^*)-learnable uniformly in majorants $\underline{a}^*$ for $\underline{a}$.**

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- The metatheorem **explains the special form** $(f_2 \circ \tilde{g} \circ f_1)^b(0)$ **of numerous metastability bounds** extracted.
- If a certain **gap condition** is satisfied by (B^*, L^*), then one gets fluctuation bounds.

Thank you Professor Troelstra!