THINKING AGAIN ABOUT CHOICE SEQUENCES

for the

Anne Troelstra Memorial Event 2020

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SOME WORKS OF ANNE TROELSTRA

1968 The Theory of Choice Sequences. In Logic Methodology and Philosophy of Science III (ed. van Rootselaar, Stahl) North-Holland

1969 Informal Theory of Choice Sequences. Studia Logica XXV

1970 (w GK) Formal Systems for some branches of Intuitionistic Analysis. Annals of Mathematical Logic 1.

(w DvD) Projections of Lawless Sequences. In Intuitionism and Proof Theory (ed Kino, Myhill, Vesley) North-Holland.

1969 Principles of Intuitionism. Springer LNM 95.

1973 Metamathematical Investigations of Intuitionistic Arithmetic and Analysis. Springer LNM 344.

1975 Non-extensional Equality. Fundamenta Mathematica 82.

1977 Choice Sequences. Clarendon Press, Oxford

MORE WORKS OF ANNE TROELSTRA

1982 On the origin and development of Brouwer's concept of choice sequence. In The LEJ Brouwer Centenary Symposium (ed Troelstra, van Dalen) North-Holland.

1983 Analysing Choice Sequences. Journal of Philosophical Logic 12.

1985 Choice Sequences and Informal Rigour. Synthese 62.

1988 (w DvD) Constructivism in Mathematics. (Two Vols) North-Holland

1992 Lectures on Linear Logic. (CLSI Lecture Notes 29)

1996 (w HS) Basic Proof Theory. Cambridge Tracts in Theoretical Computer Science 43.

1996 Choice Sequences, a retrospect. CWI Quarterly 9.

PRESERVING INTELLECTUAL INHERITANCE

in particular the idea of
Choice Sequences

Respectful Development	vs	Pious Ossification

Extreme view

Se vogliamo che tutto rimanga come è,
bisogna che tutto cambi.

BROUWER : FIRST ACT OF INTUITIONISM

- completely separates mathematics from mathematical language,
- origin in the basic phenomenon of the perception of a move of time,
- the basic operation of mathematical construction is the mental creation of the two-ity of two mathematical systems previously acquired

It is introspectively realized how this operation continually displaying unaltered retention by memory, successively generates each natural number, the infinitely proceeding sequence of the natural numbers, arbitrary finite sequences and infinitely proceeding sequences of mathematical systems previously acquired ...

BROUWER: SECOND ACT OF INTUITIONISM

The second act of intuitionism recognizes the possibility of generating new mathematical entities:

First in the form of infinitely proceeding sequences whose terms are chosen more or less freely from mathematical entities previously acquired; in such a way that the freedom existing perhaps at the first choice may be irrevocably subjected again and again to progressive restrictions at subsequent choices, while all these restricting interventions, as well as the choices themselves, may, at any stage, be made to depend on possible future mathematical experiences of the creating subject.

Secondly in the form of mathematical species ...

TROELSTRA'S CAREER

as an exemplary case of

REPECTFUL DEVELOPMENT

Based on Informal Rigour (à la Kreisel)

Axiomatic Method (modern sense)

Informal rigour: increasingly delicate analysis of CSs

Axiomatic method: elimination theorems

(a coherent project identified by
Joan Moschovakis)

AN EARLY INSIGHT

The gap between

- Spreads ~ (lawlike) closed sets Π^0_1
- Analytic sets ~ images of Brouwer operations Σ^1_1

Kreisel's axiom of spread data

$$\vdash^{\alpha} A(\alpha) \longrightarrow \exists c \in Spd \;\; \alpha \in c \wedge \forall \gamma \in c \; A(\gamma)$$

is "incompatible" with

Closure under Brouwer Operations

$$\vdash^{e} \forall \beta \, \exists \alpha \;\; \alpha = e|\beta$$

(Difficulty comes from other constructive principles.)

ABSTRACT ISSUE

Consider Brouwer operation e and consider the formula

$$\alpha \in \mathrm{Im}\, e \;\equiv\; \exists \beta \;\; \alpha = e|\beta$$

Spread Data gives

$$\alpha \in \mathrm{Im}\, e \leftrightarrow \exists c \in \mathrm{Spr} \;\; \alpha \in c \wedge c \subseteq \mathrm{Im}\, e$$

Declare 'generic' γ and set $\alpha = e|\gamma$. Then $\alpha \in \mathrm{Im}\, e$ and we get

$$\forall \gamma\, \exists c \in \mathrm{Spr} \;\; e|\gamma \in c \wedge c \subseteq \mathrm{Im}\, e.$$

(Every analytic set is a union of closed sets!)

APPLY CHOICE PRINCIPLE

AC-CF $\quad \forall \alpha \, \exists a \; A(\alpha, a) \longrightarrow \exists (b_n) \, \forall \alpha \, \exists k \; A(\alpha, b_k)$

We deduce

$$\exists (b_n \in \mathrm{Spr}) \, \forall \gamma \, \exists k \;\; \mathrm{el}|\gamma \in b_k \wedge b_k \subseteq \mathrm{Ime}.$$

(Every analytic set is a countable union of closed sets $\sim \Sigma^1_1 \equiv \Sigma^0_2$!!)

BUT NB (J. Moschovakis) With Church's Thesis, the true arithmetic hierarchy closes at Σ^0_3.

REFINEMENT : BROUWER CONTINUITY

BC-F $\qquad \forall\alpha\, \exists a\, A(\alpha, a) \longrightarrow \exists e\, \exists b_x\, \forall\alpha\, A(\alpha, b_{e(\alpha)})$

$\Longleftrightarrow \qquad$ AC-CF $\;+\;$ BC-N

$$\forall\alpha\, \exists k\, A(\alpha, k) \longrightarrow \exists e\, \forall\alpha\, A(\alpha, e(\alpha)).$$

From this we can ensure that our sequence (b_x) of spreads enumerate exactly those needed:

$$\forall x\, \exists\gamma\; e|\gamma \in b_x \wedge b_x \subseteq \mathrm{Im}\, e,$$

as well as

$$\forall\gamma\, \exists k\; e|\gamma \in b_k \wedge b_k \subseteq \mathrm{Im}\, e.$$

So

$$\mathrm{Im}\, e = \bigcup \{ b_x \mid x \in \mathbb{N} \}.$$

TROELSTRA'S EXAMPLE I

Take a lawlike $a : \mathbb{N} \times \mathbb{N} \to 2$ or $a : \mathbb{N} \to 2^{\mathbb{N}}$ or $(a_x \in 2^{\mathbb{N}})$ a sequence of decidable subsets of $\mathbb{N}$.

Start by constructing a very simple spread.

Set $a_x^+(0) = 1 \quad a_x^+(n+1) = a_x(n)$ and define S_a by

$$u \in S_a \iff \forall i < |u| \; a_i^+(u_i) = 1$$

(So the branching is fixed at each level.)

OBSERVE

$$\forall x \underbrace{\exists n \; a_x(n) = 1}_{(a_x \text{ inhabited})} \iff \exists \alpha \in S_a . \forall i \; \alpha(i) \neq 0$$

TROELSTRA'S EXAMPLE II

(minimal use of Brouwer Operations)

1) For any spread S there is a Brouwer e_S with $S = \mathrm{Im}\, e_S$.

2) There is a Brouwer operation sgn collapsing $\mathbb{N}^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ in the obvious way.

3) Brouwer operations compose.

So we can set $e = \mathrm{sgn} | e_{S_a}$.

OBSERVE

$$\forall x \; \exists n \; a_x(n) = 1 \iff \dot{1} = \lambda x.1 \in \mathrm{Im}\, e.$$

TROELSTRA'S EXAMPLE III

Earlier we saw that for any Brouwer e we can construct a sequence (b_x) of spreads such that

$$\mathrm{Im}\, e = \bigcup \{ b_x \mid x \in \mathbb{N} \}.$$

So $\; 1 \in \mathrm{Im}\, e \iff \exists x \; 1 \in b_x \iff \exists x \, \forall m \; b_x(1^m) = 1$.

So

$$\forall x \, \exists n \; a_x(n) = 1 \iff \exists x \, \forall m \; b_x(1^m) = 1$$

Setting $\bar{b}_x(m) = b_x(1^m)$ we have shown

$$\forall a \, \exists \bar{b} \; \left(\forall x \, \exists n \; a(x,n) = 1 \leftrightarrow \exists x \, \forall m \; \bar{b}(x,m) = 1 \right)$$

Incompatible with Church's Thesis.

A MODEST PROPOSAL

First act of intuitionism

$\rightsquigarrow$ Inductive Types (W-Types)

Second act of intuitionism

$\rightsquigarrow$ Coinductive Types (Haskell)

Brouwer's explanations already indicate: mathematical activity combines these.

CS from this perspective?

BROUWER OPERATIONS

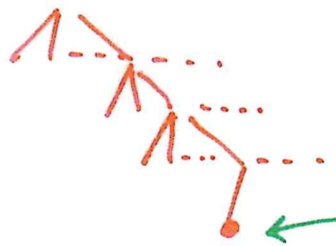
Given A, B define $K_A B$ by

$$\frac{b \in B}{[b] \in K_A B} \qquad \frac{s_a \in K_A B \quad [a \in A]}{\langle \lambda a . s_a \rangle \in K_A B}$$

~ the initial algebra

$$K_A B \xleftarrow{\ \cong\ } B + K_A B^A$$

~ collection of well-founded trees



APPLICATION (INDUCTIVE DEFINITION)

$$\mathrm{act} : K_A B \times A^{\mathbb{N}} \longrightarrow B$$

$$\mathrm{act}([b], \alpha) = b$$

$$\mathrm{act}(\langle \lambda a . S_a \rangle, \alpha) = \mathrm{act}(S_{\alpha(0)}, \alpha')$$

coming from evident algebra structure

$$(A^{\mathbb{N}} \Rightarrow B) \longleftarrow B + (A^{\mathbb{N}} \Rightarrow B)^A$$

REMARK This uses the coalgebra structure

$$A^{\mathbb{N}} \longrightarrow A \times A^{\mathbb{N}}.$$

FUNCTIONS TO FUNCTIONS

(Traditional representation)

We have

$$K_A B^{\mathbb{N}} \times A^{\mathbb{N}} \longrightarrow (K_A B \times A^{\mathbb{N}})^{\mathbb{N}} \longrightarrow B^{\mathbb{N}}$$

Then taking $A = \mathbb{N}$ and using $B \to K_{\mathbb{N}} B \to (K_{\mathbb{N}} B)^{\mathbb{N}}$

we have

$$K_{\mathbb{N}} B \xrightarrow{\simeq} B + (K_{\mathbb{N}} B)^{\mathbb{N}} \longrightarrow (K_{\mathbb{N}} B)^{\mathbb{N}}$$

So that $(K_{\mathbb{N}} B)^{\mathbb{N}} \lhd K_{\mathbb{N}} B$ and we have

an action

$$K_{\mathbb{N}} B \times \mathbb{N}^{\mathbb{N}} \longrightarrow B^{\mathbb{N}}$$

(As in Kreisel Troelstra.)

COMPOSITION OF BROUWER OPERATIONS

We can do this with representations $K_A B^{\mathbb{N}}$,

BUT problems with identity

associativity

CURE take a retract $L_A B \lhd K_A B^{\mathbb{N}}$

MORAL $K_A B^{\mathbb{N}}$ is the wrong coinductive type.

HANCOCK PATTINSON GHANI

2009 Representation of Stream Processes using Nested Fixed Points. Logical Methods in Computer Science, 5.

Recall $K_A B = \mu X.\ B + X^A$.

Define $L_A B = \nu Y.\ K_A(B \times Y) = \nu Y\, \mu X\, (B \times Y + X^A)$.

Have

$$\begin{array}{ccc} L_A B & \xrightarrow{\cong} & K_A(B \times L_A B) & \\ & & \uparrow{\cong} & \\ & & B \times L_A B + K_A(B \times L_A B)^A & \xleftarrow{\cong} B \times L_A B + L_A B^A . \end{array}$$

Any element of $L_A B$ is either $[b, s]$ $(s \in L_A B)$

or $\langle \lambda a.\, s_a \rangle$ $(s_a \in L_A B\ [a \in A])$.

We use this in equations but it gives no universal property.

APPLICATION (COINDUCTIVE DEFINITION)

$$\underline{\mathrm{act}} : L_A B \times A^{\mathbb{N}} \longrightarrow B^{\mathbb{N}}$$

$$\underline{\mathrm{act}}([b, s], \alpha) = b . \underline{\mathrm{act}}(s, \alpha)$$

$$\underline{\mathrm{act}}(\langle \lambda a. s_a \rangle, \alpha) = \underline{\mathrm{act}}(s_{\alpha(0)}, \alpha').$$

Why does this make sense?

- Use $B^{\mathbb{N}} \longrightarrow B \times B^{\mathbb{N}}$ a final coalgebra so seek

$$L_A B \times A^{\mathbb{N}} \longrightarrow B \times L_A B \times A^{\mathbb{N}}$$

- Equivalently

$$K_A(B \times L_A B) \times A^{\mathbb{N}} \longrightarrow B \times L_A B \times A^{\mathbb{N}}$$

APPLICATION CONTINUED

We tweak our old $act : K_A C \times A^{\mathbb{N}} \longrightarrow C$ to obtain

$$act^* : K_A C \times A^{\mathbb{N}} \longrightarrow C \times A^{\mathbb{N}}$$

$$act^*([c], \alpha) = (c, \alpha)$$

$$act^*(\langle \lambda a . s_a \rangle, \alpha) = act^*(s_{\alpha(0)}, \alpha')$$

And this comes from the algebra structure

$$(A^{\mathbb{N}} \Rightarrow C \times A^{\mathbb{N}}) \longleftarrow C + (A^{\mathbb{N}} \Rightarrow C \times A^{\mathbb{N}})^{N}$$

MORAL HERE ?

DETOUR ON SPREADS

Let $D^+(\mathbb{N})$ be the inhabited decidable subsets of $\mathbb{N}$.

(So $D^+\mathbb{N} \cong \mathbb{N} \times 2^{\mathbb{N}}$.)

Coinductive definition

$$\mathrm{Sprd} \xrightarrow{\cong} \sum I \in D^+(\mathbb{N})\ \mathrm{Spr}^I$$

Traditional coding $\chi : \mathrm{Sprd} \longrightarrow (\mathrm{List}\,\mathbb{N} \Rightarrow 2)$

$$\chi([I, (s_i)_{i \in I}], u) = \begin{cases} \text{if } u = \langle\rangle \text{ then } 1 \\ \text{else } u = u_0.u' \text{ and} \\ \quad \text{if } u_0 \in I \text{ then } \chi(s_{u_0}, u') \\ \quad \text{else } 0 \,. \end{cases}$$

SPREAD MEMBERSHIP

$$\alpha \in [I, (S_i)] \iff \alpha(0) \in I \wedge \alpha' \in S_{\alpha(0)}.$$

What could that mean?

Define coinductively

$$E \xrightarrow{\cong} \Sigma I \in D^+\mathbb{N}.\ I \times E^I$$

Then have $\quad E \xrightarrow{\mathrm{fst}} \mathbb{N}^{\mathbb{N}}$

from $\quad E \longrightarrow \Sigma I \in D^+\mathbb{N}\ I \times E^I \longrightarrow \mathbb{N} \times E$

and $\quad E \xrightarrow{\mathrm{snd}} \mathrm{Spread}$

from $\quad E \longrightarrow \Sigma I \in D^+\mathbb{N}\ I \times E^I \longrightarrow \Sigma_{I \in D^+\mathbb{N}}\ E^I$

Why are these jointly monic!

BROUWER CATEGORY

Objects $\quad A, B, C$

Maps $\quad L_A B$

Identity $\quad id_A \in L_A A \quad : \quad id_A = \langle \lambda a. [a, id_A] \rangle$

Composition $\quad L_B C \times L_A B \longrightarrow L_A C$

$$[c, t] \circ s = [c, t \circ s]$$

$$\langle \lambda b. t_b \rangle \circ [\bar{b}, \bar{s}] = t_{\bar{b}} \circ \bar{s}$$

$$\langle - \rangle \circ \langle \lambda a. s_a \rangle = \langle \lambda a. \langle - \rangle \circ s_a \rangle .$$

BROUWER TOPOLOGY

For each A canonical covers are determined by the elements of $K_A 1$.

Each such determines a set of nodes in $\mathrm{List}\,A$ and each node u determines an idempotent e_u

$$e_{\langle\rangle} = id \qquad e_{u_0 . u'} = \langle \lambda a . [u_0, e_{u'}] \rangle$$

Cover is given by the maps factoring through these.

YET ANOTHER CS MODEL

Use the monoid $L_{\mathbb{N}}\mathbb{N}$ with the induced topology in the style of Fourman

van der Hoeven - Moerdijk

MAIN FEATURE

The representable does not give the internal $\mathbb{N}^{\mathbb{N}}$, cf vdH-M on spread data. But this time it is "big" and extensionality fails.

TROELSTRA : INTENSIONAL EQUALITY

1975 Non-extensional equality

1996 Choice Sequences: a retrospect

From 1996 I follow another analytic approach which highlights the "intensional" aspects of choice sequences.

My last meeting with Anne Troelstra at the Utrecht Topology Feest 2018 (ForIM)

Our discussion
- intensional aspects of CS
- history of botany