Patterns and Principles in Troelstra's Metamathematical Work

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Perhaps more than anyone else in the last 50 years, Anne Troelstra had a comprehensive, encyclopedic knowledge of the classical and constructive metamathematics of intuitionistic formal systems. He shared this knowledge generously, by teaching and by writing many influential papers and books which weave his original work into a rich context of history and the contributions of others.

His own metamathematical work seemed to follow a **pattern**:

- Select a domain of discourse (e.g. arithmetic or analysis).
- Identify and analyze the key concepts from an intuitionistic viewpoint, using informal rigor.
- Formulate axioms for basic concepts and important models.
- Build corresponding formal systems, with intuitionistic logic.
- Compare these formal systems as constructively as possible.

This pattern emerged early and led to many nice theorems.

Among his best known results are

- the elimination of choice sequence variables,
- the axiomatic characterization of realizability interpretations,
- conservative extension theorems and admissible rules.

There is a deceptively simple **pattern** to all the proofs:

Find intuitionistic systems $\mathbf{S} \subsetneq \mathbf{T}$ and a syntactic translation τ of formulas E of the language $\mathcal{L}(\mathbf{T})$ to formulas $\tau(E)$ of $\mathcal{L}(\mathbf{S})$ so that

(i) $\vdash_{\mathbf{T}} (E \leftrightarrow \tau(E))$ and

(ii) $\vdash_{\mathbf{T}} E \;\Leftrightarrow\; \vdash_{\mathbf{S}} \tau(E)$ can be proved finitistically.

$\star$ If $\mathcal{L}(\mathbf{T})$ has choice sequence variables but $\mathcal{L}(\mathbf{S})$ does not, then choice sequence variables are eliminated from sentences of $\mathcal{L}(\mathbf{T})$.

$\star$ If $\tau(E)$ is $\exists f(f\,\mathbf{r}E)$ where $f\,\mathbf{r}E$ expresses "f realizes E," then $\mathbf{T}$ precisely characterizes realizability over $\mathbf{S}$.

$\star$ If $\tau(0=1)$ is $(0=1)$, then $\mathbf{T}$ is consistent relative to $\mathbf{S}$.

$\star$ If $\tau(E) = E$ for E in $\mathcal{L}(\mathbf{S})$, $\mathbf{T}$ is a conservative extension of $\mathbf{S}$.

Anne chose fruitful combinations of $\mathbf{S}$, $\mathbf{T}$ and τ, often inventing a **principle** Γ so the pattern would work with $\mathbf{T} = \mathbf{S} + \Gamma$.

Consider the axiomatization of realizability. Given a classically and intuitionistically correct system $\mathbf{S}$ in which a nonclassical system $\mathbf{S}^+$ extending $\mathbf{S}$ can be interpreted, e.g. by Kleene's q-realizability or Kreisel's m-realizability. Given that

- $\tau(E)$ expresses "E is true under the interpretation," and
- $\vdash_{\mathbf{S}^+} E \Rightarrow\ \vdash_{\mathbf{S}} \tau(E)$ for sentences E of $\mathcal{L}(\mathbf{S})$, but
- the converse is known to be false.

The challenge is to find the right axiom(s) or schema(s) Γ so that

(i) $\vdash_{\mathbf{T}} (E \leftrightarrow \tau(E))$ and

(ii) $\vdash_{\mathbf{T}} E \Leftrightarrow\ \vdash_{\mathbf{S}} \tau(E)$

where $\mathbf{T} = \mathbf{S} + \Gamma \supset \mathbf{S}^+ \supset \mathbf{S}$.

Then Γ "axiomatizes" the interpretation of $\mathbf{S}^+$ over $\mathbf{S}$.

Troelstra's focus on intuitionistic formal systems goes back to his undergraduate and graduate work with Arend Heyting, who formalized Brouwer's informal intuitionistic logic and arithmetic.

As a postdoctoral scholar in 1966-67, Anne discovered an error in Kreisel's axioms for lawless sequences. Kreisel (1968) proved the *first elimination theorem* ((i) of the pattern) for the theory **LS** of lawless sequences. Troelstra (1969) proved the *second elimination theorem* ((ii) of the pattern) for the corrected **LS** over its neutral lawlike subsystem **IDK**. Moreover, **LS** is conservative over **IDK**.

Lawless sequences are extremely antisocial. M. Hyland showed how Troelstra corrected Kreisel's axioms for choice sequences given by a spread, resulting in a theory **CS** of choice sequences closed under lawlike continuous operations. First and second elimination theorems for **CS** over $\mathbf{IDB}_1$ ($\equiv$ **IDK**) are in Kreisel and Troelstra (1970). Troelstra (1971) also proved that **CS** is a conservative extension of a notational variant of the formal system **I** of Kleene and Vesley's "Foundations of Intuitionistic Mathematics" (1965).

By reflecting on Kleene's, Kreisel's and Vesley's work, Troelstra discovered axiomatic characterizations of a variety of realizability interpretations for formal systems based on intuitionistic logic (the *"axiomatization of realizability"*). All these characterizations follow the pattern described, with a first and second translation theorem. Each characterization involves a new mathematical principle of independent interest; some justify admissible, nonderivable rules.

Let's look first at some details of the elimination of choice sequence variables, and then at the axiomatization of realizability. Along the way, we'll observe Troelstra's principles in their natural environments.

Brouwer's notion of choice sequence

Already in 1907 Brouwer recognized the impossibility of building a continuum using only lawlike fundamental sequences of rationals. In 1908 he rejected the universal law of excluded middle (the "First Act of Intuitionism"), and by 1918 he was developing the notions of *spread*, *choice sequence* and *species*. In "Historical background, principles and methods of intuitionism" (1952) he wrote that the "Second Act of Intuitionism" explicitly recognizes "the possibility of generating new mathematical entities:

"firstly in the form of infinitely proceeding sequences $p_1, p_2, \ldots$ whose terms are chosen more or less freely from mathematical entities previously acquired . . . ;

"secondly in the form of mathematical species, i.e. properties supposable for mathematical entities previously acquired, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it . . . "

Kreisel's lawless sequences, as improved by Troelstra

Kreisel's idea of a lawless (originally, "completely free") sequence was diametrically opposed to Brouwer's notion of a lawlike sequence or "sharp arrow." A lawless sequence of natural numbers was a choice sequence admitting no restrictions; at each stage of its generation, every natural number was eligible to be chosen. Kreisel's **LS** and Troelstra's **CS** are three-sorted intuitionistic formal theories with variables x,y,z,... over numbers, a,b,c,... over lawlike sequences, and $\alpha, \beta, \gamma, \ldots$ over lawless sequences.

LS $=$ **IDB**$_1$ $+$ LS1-4   and

CS $=$ **IDB**$_1$ $+$ GC1-4, where

IDB$_1$ $=$ **EL** $+$ K1-3 $+$ AC$_{01}$, where

EL ("elementary analysis") is a two-sorted "lawlike" extension of Heyting arithmetic **HA** with primitive recursive function constants, λ-abstraction, and countable choice for quantifier-free relations.

EL codes finite sequences of natural numbers primitive recursively. Every natural number n codes a unique sequence whose length $\leq n$ is recoverable from n. $\langle\,\rangle = 0$ codes the empty sequence; $\langle x\rangle$ codes the sequence consisting of just x; and $*$ denotes concatenation. $\mathbf{IDB}_1$ adds to the language of **EL** a constant K representing the inductively generated class of lawlike neighborhood functions of continuous functionals of type 1, and adds to the axioms of **EL**

K1. $K(\lambda n.(x+1))$.

K2. $a(\langle\,\rangle) = 0 \;\&\; \forall x\, K(\lambda n.a(\langle x\rangle * n)) \rightarrow K(a)$.

K3. $\forall a\,(A(Q,a) \rightarrow Q(a)) \rightarrow \forall a\,(K(a) \rightarrow Q(a))$
for all formulas Q of the language, where
$A(Q,a) \equiv \exists x(a = \lambda n.x+1) \vee (a(0) = 0 \;\&\; \forall x Q(\lambda n.a(\langle x\rangle * n)))$.

$\mathbf{IDB}_1$ also has an axiom schema of countable choice:

AC_{01}. $\forall x \exists a\, A(x,a) \rightarrow \exists b \forall x\, A(x,(b)_x)$.

Here $(b)_x = \lambda y.b(j(x,y))$ where j is a constant of **EL** representing a primitive recursive pairing function.

The formal theory **LS** adds to the language of $\mathbf{IDB}_1$ variables $\alpha, \beta, \gamma, \ldots$ for, and quantifiers $\forall\alpha, \exists\alpha \ldots$ over, lawless sequences. There are two new axioms and two new axiom schemas.

L1. $\forall n \exists \alpha (\alpha \in n)$ is the *density* axiom.

The next axiom says that (not only intensional, but also) *extensional* equality of lawless sequences is decidable.

L2. $\forall \alpha \forall \beta (\alpha \neq \beta \vee \alpha = \beta)$.

To express *relative independence* of lawless variables Troelstra defined quantifiers $\dot{\forall}\alpha, \dot{\exists}\alpha$ so that e.g. $\dot{\forall}\alpha A(\alpha, \beta, \gamma)$ is equivalent to $\forall \alpha (\alpha \neq \beta \ \& \ \alpha \neq \gamma \rightarrow A(\alpha, \beta, \gamma))$, and if $\vec{\alpha} = \alpha_0, \ldots, \alpha_k$ then $\dot{\forall}\vec{\alpha} A(\vec{\alpha})$ expresses $\forall \alpha_0 \ldots \forall \alpha_k (\forall i < j \leq k (\alpha_i \neq \alpha_j) \rightarrow A(\vec{\alpha}))$.

With all lawless parameters shown, the schema of *open data* is

L3. $\dot{\forall}\alpha (A(\alpha, \vec{\beta}) \rightarrow \exists n (\alpha \in n \ \& \ \dot{\forall}\gamma \in n \, A(\gamma, \vec{\beta})))$

and the *bar continuity* schema (with lawlike e and b) is

L4. $\dot{\forall}\vec{\alpha} \exists b A(\vec{\alpha}, b) \rightarrow \exists e (K(e) \ \& \ \forall n (e(n) \neq 0 \rightarrow \exists b \dot{\forall}\vec{\alpha} \in n A(\vec{\alpha}, b)))$

Kreisel and Troelstra's "elimination of lawless sequences"

In (1968) Kreisel proved the *"first elimination theorem"* for **LS**: Any formula with no free lawless sequence variables is equivalent to one without lawless sequence variables. In (1969) Troelstra stated the *"second elimination theorem"* for **LS** over $\mathbf{IDB_1}$. For a clear exposition of the proof see Volume II of Troelstra and van Dalen's "Constructivism in Mathematics: An Introduction" (1988).

The elimination of lawless sequences holds for **LS** in $\mathbf{IDB}_1$: There is a syntactic translation τ mapping each formula E of $\mathcal{L}(\mathbf{LS})$ without free lawless sequence variables to a formula $\tau(E)$ without any lawless sequence variables such that

(i) $\vdash_{\mathbf{LS}} (E \leftrightarrow \tau(E))$.

(ii) $\vdash_{\mathbf{LS}} E \Leftrightarrow \vdash_{\mathbf{IDB_1}} \tau(E)$.

(iii) $\tau(E) \equiv E$ if E has no lawless sequence variables.

Corollary: **LS** is a conservative extension of $\mathbf{IDB}_!$.

The translation τ which gradually eliminates quantifiers over lawless sequences from formulas in **LS** is complex. It involves e.g.

- using LS2 to replace $\forall\alpha A(\alpha,\beta)$ by $(A(\beta,\beta)\ \&\ \dot{\forall}\alpha A(\alpha,\beta))$,
- using LS2 to replace $\exists\alpha A(\alpha,\beta)$ by $(A(\beta,\beta) \vee \dot{\exists}\alpha A(\alpha,\beta))$,
- replacing $A \vee B$ by $\exists n((n=0 \to A)\ \&\ (n \neq 0 \to B))$,
- replacing $\dot{\exists}\alpha A(\alpha,\beta)$ by $\exists n\dot{\forall}\alpha \in n\, A(\alpha,\beta)$ using LS3,
- moving $\dot{\forall}\alpha \in n$ to the inside using LS4, which introduces new number and lawlike sequence quantifiers only, and
- replacing $\dot{\forall}\alpha \in n(s(\alpha) = t(\alpha))$ by $\forall a \in n(s(a) = t(a))$.

As an example, consider LS1: $\forall n\exists\alpha(\alpha \in n)$, where $\alpha \in n$ is a prime formula expressing that n codes an initial segment of α.

By the algorithm (simplified as LS1 has no lawless parameters),

$\tau(\text{LS1}) = \tau(\forall n\exists m\forall\alpha \in m\,(\alpha \in n)) = \forall n\exists m\forall a \in m\,(a \in n)$

which is provable in $\mathbf{IDB}_1$ (or even in **EL**), and is equivalent to $\forall n\exists m\forall\alpha \in m\,(\alpha \in n)$ in **LS** by an argument involving LS4.

Troelstra's extension principle; Brouwer's Bar Theorem

$\mathbf{IDB}_1$ has lawlike sequence and number variables, and the class K of neighborhood functions defined inductively by axioms K1-3. If

$K_0(e) \equiv \forall a \exists n\, e(\overline{a}(n)) \neq 0 \;\&\; \forall m \forall n (e(n) > 0 \rightarrow e(n) = e(n * m))$

then $\vdash_{\mathbf{IDB_1}} \forall e(K(e) \rightarrow K_0(e))$, but not conversely.

Brouwer's Bar Theorem for lawlike sequences could be expressed by $\forall e(K_0(e) \rightarrow K(e))$, but Kleene's recursive counterexample argues against it. So $\nvdash_{\mathbf{IDB_1}} \forall e(K_0(e) \rightarrow K(e))$. In contrast,

$\vdash_{\mathbf{LS}} \forall e(K(e) \rightarrow \forall \alpha \exists x\, e(\overline{\alpha}(x)) \neq 0)$, and if $K_0^*(e)$ is like $K(e)$ but with α in place of a then $\vdash_{\mathbf{LS}} \forall e(K_0^*(e) \rightarrow K(e))$ using LS4.

Troelstra argued that the initial segments of *any* choice sequence can be viewed as initial segments of a lawless sequence in process of generation. In Brouwer's terminology, **Troelstra's extension principle** states that every bar on the lawless sequences bars *all* sequences of natural numbers, so the Bar Theorem holds for **LS**.

General choice sequences: the principle of analytic data

An alternate universe of choice sequences closed under lawlike continuous operations is described by $\mathbf{CS} = \mathbf{IDB}_1 +$ GC 1-4, with the same language as **LS** but a very different interpretation. Let $e|\alpha = \beta$ abbreviate $\forall y(\lambda n.e(\langle y \rangle * n))(\alpha) = \beta(y))$ where $e(\alpha) = t$ abbreviates $\exists x\, e(\overline{\alpha}(x)) = t + 1$. The new axioms are

GC1. $\forall e(K(e) \rightarrow \forall \alpha \exists \beta (e|\alpha = \beta))$ and $\forall \alpha \forall \beta \exists \gamma (j(\alpha, \beta) = \gamma)$.

GC2. $A(\alpha) \rightarrow \exists e(K(e)\ \&\ \exists \beta(e|\beta = \alpha)\ \&\ \forall \beta A(e|\beta))$ is the **principle of analytic data**. There are two *continuity axioms*:

GC3. $\forall \alpha \exists b A(\alpha, b) \rightarrow \exists e \forall n(e(n) \neq 0 \rightarrow \exists b \forall \alpha \in n A(\alpha, b))$.

GC4. $\forall \alpha \exists \beta\, A(\alpha, \beta) \rightarrow \exists e \forall \alpha\, A(\alpha, e|\alpha)$ (like Kleene's continuous choice but with a lawlike modulus).

- **Elimination of choice sequences** holds for **CS** in $\mathbf{IDB}_1$,
- $\vdash_{\mathbf{CS}} \forall e(K_0^*(e) \leftrightarrow K(e))$ (the monotone bar theorem), and
- $\vdash_{\mathbf{CS}} \forall \alpha \neg\neg \exists b(\alpha = b)$. (In contrast, $\vdash_{\mathbf{LS}} \forall \alpha \neg \exists b(\alpha = b)$.)

Kleene's function-realizability; almost negative formulas

Kleene's formal system **I** of intuitionistic analysis is two-sorted, with variables $a, b, c, \ldots, x, y, z, \ldots$ over natural numbers and $\alpha, \beta, \gamma, \ldots$ over arbitrary choice sequences. When Myhill objected to the absence of lawlike sequence variables, Kleene responded that the general recursive functions could be coded by numbers.

Kleene defined a function-realizability interpretation for **I**, proving

- if $\Gamma \vdash_{\mathbf{I}} E$ where Γ are recursively realizable, so is E, and
- $0 = 1$ is not recursively realizable, so $\nvdash_{\mathbf{I}} \; 0 = 1$,

using arguments formalizable in the classically correct subsystem **B** of **I**. Kleene (1969) completed the formalization and proved e.g.

- $\vdash_{\mathbf{I}} E \; \Rightarrow \vdash_{\mathbf{B}} \exists\gamma(GR(\gamma) \;\&\; \gamma \, \mathbf{rf} \, E)$.
- (E is closed and $\vdash_{\mathbf{I}} E) \Rightarrow (\vdash_{\mathbf{B}} (\{\mathbf{n}\}\downarrow \;\&\; \{\mathbf{n}\}\mathbf{rf}\,E)$ for some $\mathbf{n}$).
- $\vdash_{\mathbf{B}} \; (A \leftrightarrow \exists\beta(\beta \, \mathbf{rf} \, A))$ for *almost negative* formulas A (no $\vee$, and no $\exists$ except immediately in front of a prime formula).

Troelstra's Generalized Continuity principle GC

Kleene (1969) formalized the theory of recursive partial functions in a subsystem **M** of **B** with countable choice AC_{01} weakened to countable comprehension AC_{00}! In an aside, Troelstra (1973) observed that countable comprehension for quantifier-free relations suffices, so this part of Kleene's formalization is available in **EL**.

Anne saw something else Kleene evidently missed. He observed

- The formula $\gamma\,\mathbf{rf}\,E$ is inductively defined and almost negative.
- The only obstruction to an inductive proof in Kleene's **I** of $(E \leftrightarrow \exists\gamma\,(\gamma\,\mathbf{rf}\,E))$ for *all* formulas E is the clause for $\rightarrow$.
- That obstruction disappears if Kleene's continuous choice principle CC_{11} is extended to *any almost negative hypothesis*.

Troelstra's Generalized Continuity principle GC is

$$\forall\alpha[A(\alpha) \rightarrow \exists\beta B(\alpha,\beta)] \rightarrow \exists\gamma\forall\alpha[A(\alpha) \rightarrow \exists\beta[(\gamma|\alpha = \beta)\&B(\alpha,\beta)]]$$

where $A(\alpha)$ must be almost negative. Note that $\vdash_{\mathbf{EL}+GC} \mathrm{CC}_{11}$.

Troelstra's "Extended Church's Thesis" ECT_0

There is a strong parallel between intuitionistic analysis and constructive recursive mathematics **CRM**, which adds at least Markov's Principle MP and "Church's Thesis" CT_0 to intuitionistic arithmetic. **HA** + MP is classically correct but the recursive choice principle CT_0 is not. The consistency of **HA** + MP + CT_0 was established by Kleene and David Nelson, using Kleene's (1945) number-realizability which Nelson formalized in his dissertation.

- If Γ are realizable and $\Gamma \vdash_{\mathbf{HA}} E$, then E is realizable.
- $\vdash_{\mathbf{HA}+CT_0} E \Rightarrow \vdash_{\mathbf{HA}} \exists f\,(f\,\mathbf{rn}\,E)$, but not conversely.
- $\vdash_{\mathbf{HA}} \neg\exists f\,(f\,\mathbf{rn}\,(0=1))$.
- $\vdash_{\mathbf{HA}} (A \leftrightarrow \exists f(f\,\mathbf{rn} A))$ if A is *almost negative*.

Anne saw what was missing. **Extended Church's Thesis** ECT_0 is
$\forall x[A(x) \rightarrow \exists y B(x,y)] \rightarrow \exists f \forall x[A(x) \rightarrow \{f\}(x)\downarrow \,\&\, B(x,\{f\}(x))]$
where $A(x)$ must be almost negative. Note that $\vdash_{\mathbf{HA}+ECT_0} \mathrm{CT}_0$.

Troelstra's axiomatic characterizations of realizability

Two examples of his (and van Oosten's) many characterizations of realizability and modified realizability (cf. $\mathbf{HA}^\omega$, **HRO**, **HEO**,...):

Axiomatization of Kleene's number-realizability (Troelstra):

(i) $\vdash_{\mathbf{HA}+ECT_0} (E \leftrightarrow \exists g\,(g\,\mathbf{rn}\,E))$ for all formulas E of $\mathcal{L}(\mathbf{HA})$.

(ii) If Δ are closed formulas of $\mathcal{L}(\mathbf{HA})$ and $\Delta \vdash_{\mathbf{HA}} \exists f(f\ \mathbf{rn}\ A)$ for each $A \in \Delta$, then $\Delta \vdash_{\mathbf{HA}+ECT_0} E \;\Leftrightarrow\; \Delta \vdash_{\mathbf{HA}} \exists g\,(g\,\mathbf{rn}\,E)$.

(iii) If also E is closed and if to each $A \in \Delta$ there is a numeral $\mathbf{f}$ such that $\Delta \vdash_{\mathbf{HA}} (\mathbf{f}\ \mathbf{rn}\ A)$, then there is a numeral $\mathbf{g}$ such that $(\Delta \vdash_{\mathbf{HA}+ECT_0} E) \Leftrightarrow (\Delta \vdash_{\mathbf{HA}} (\mathbf{g}\ \mathbf{rn}\ E))$.

Corollary. $\vdash_{\mathbf{HA}+MP+ECT_0} E \;\Leftrightarrow \vdash_{\mathbf{HA}+MP} (\mathbf{g}\,\mathbf{rn}\,E)$ for some $\mathbf{g}$.

Axiomatization of Kleene's function-realizability (Troelstra):

(i) $\vdash_{\mathbf{EL}+GC} (E \leftrightarrow \exists\gamma\,(\gamma\,\mathbf{rf}\ E))$ for all formulas E of $\mathcal{L}(\mathbf{EL})$.

(ii) If Δ consists of formulas of $\mathcal{L}(\mathbf{EL})$ such that $\Delta \vdash_{\mathbf{EL}} \exists\alpha(\alpha\mathbf{rf}A)$ for each $A \in \Delta$, then $\Delta \vdash_{\mathbf{EL}+GC} E \;\Leftrightarrow\; \Delta \vdash_{\mathbf{EL}} \exists\gamma\,(\gamma\,\mathbf{rf}\ E)$.

Heyting's arithmetic of species; the Uniformity Principle

Heyting's arithmetic of species **HAS** is a second-order intuitionistic formal system extending first-order Heyting arithmetic **HA**. It has variables $m, n, x, y, z, \ldots$ over natural numbers and $X, Y, Z, \ldots$ over species, and an axiom schema of full comprehension:

CA. $\exists X \forall x (X(x) \leftrightarrow A(x, \vec{y}, \vec{Z}))$.

Troelstra's Uniformity Principle for numbers is the schema

UP_0. $\forall X \exists n\, A(X, n) \to \exists n \forall X A(X, n)$.

Troelstra proved

- $\nvdash_{\mathbf{HAS}} \mathrm{UP}_0$ (although the corresponding *rule* is admissible).
- $\mathbf{HAS} + \mathrm{ECT}_0 + \mathrm{UP}_0 + \mathrm{MP}$ is consistent.
- A weaker version $\mathrm{UP}_0!$ $\forall X \exists! n\, A(X, n) \to \exists n \forall X A(X, n)$ "is easily proved on the assumption of $\neg\forall P(\neg P \vee \neg\neg P)$," which expresses the denial of the *principle of testability*.

($\exists! n Y(n)$ abbreviates $\exists n Y(n)$ & $\forall n \forall m (X(n)$ & $X(m) \to n = m)$.)

Formalized realizability-plus-truth; admissible rules

Anne's chapter on realizability in the Handbook of Proof Theory uses the logic of partial terms efficiently to express application ($e * x \simeq \{e\}(x)$ for **HA** and **HAS**; $e|a$ and $e(a)$ for $\mathbf{IDB}_1$; $e|\alpha$ and $e(\alpha)$ for **EL**) and the appropriate realizability predicate in a conservative extension $\mathbf{S}^*$ of each system **S**. This is possible because realizing objects are *codes* of partial function(al)s, not the functions themselves, so are of the same type as objects of $\mathcal{L}(\mathbf{S})$. The realizing objects form a partial combinatory algebra, so Anne defined realizability in a one-sorted theory **APP** with the partial recursive operations and partial continuous functions as models.

Kleene's realizability-plus-truth alters the inductive clauses for $\to$ and $\exists$, e.g. in Troelstra's formalization of **rnt** for $\mathbf{HA}^*$:

- $f\ \mathbf{rnt}\ (A \to B) \equiv \forall g((g\ \mathbf{rnt}\ A)\ \&\ A \to f * g\ \mathbf{rnt}\ B)$
- $f\ \mathbf{rnt}\ \exists x A(x) \equiv A(j_0(f))\ \&\ j_1(f)\ \mathbf{rnt}\ A(j_0(f))$.

Nelson formalized **rnt**-realizability and showed e.g. that **HA** satisfies *Church's Rule*: If $\forall x \exists y A(x,y)$ is closed, then

- $\vdash_{\mathbf{HA}} \forall x \exists y A(x,y) \Rightarrow \vdash_{\mathbf{HA}} \exists f \forall x (\{f\}(x)\downarrow \ \& \ A(x,\{f\}(x)))$.

Kleene formalized **rft**-realizability to show e.g. that **B** and **I** satisfy the *Church-Kleene Rule*:

- If $\forall \alpha \exists \beta A(\alpha,\beta)$ is closed and $\vdash_{\mathbf{I}} \forall \alpha \exists \beta A(\alpha,\beta)$, then for some n: $\vdash_{\mathbf{I}} \forall x \{\mathbf{n}\}(x)\downarrow \ \& \ \forall \alpha (\{\mathbf{n}\}|\alpha = \beta \to A(\alpha,\beta))$.

Troelstra used formalized **rnt**- and **rft**-realizability to show that

- **HA*** and **EL*** are closed under **Extended Church's Rule** : For $A(x)$ almost negative: if $\forall x[A(x) \to \exists y B(x,y)]$ is a closed theorem, so is $\exists z \forall x[A(x) \to \{z\}(x)\downarrow \ \& \ B(x,\{z\}(x))]$.
- **EL*** is also closed under a **Generalized Continuity Rule**.
- **HAS*** is closed under **Troelstra's Uniformity Rule**:

  $$\vdash_{\mathbf{HAS}^*} \forall X \exists y\, A(X,y) \ \Rightarrow \vdash_{\mathbf{HAS}^*} \exists y \forall X\, A(X,y)$$

  So quantifying over species is unlikely to be useful; number and choice sequence quantifiers should suffice.

Two applications, and a conjecture by Troelstra

Troelstra and van Dalen (1988) identify constructive recursive mathematics **CRM** with **HA** + MP + ECT_0. In **CRM** they prove the *Kreisel-Lacombe-Shoenfield-Tsejtin Theorem*: Every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous. Their proof uses the fact that "x is a gödel number of a recursive Cauchy real" is almost negative.

$A \subseteq \mathbb{N}^{\mathbb{N}}$ is a *domain of continuity* if every partial function defined at least on A is continuous on A. $B \subseteq \mathbb{N}$ is a *Church domain* if every partial function defined at least on B is recursive on B.
Theorem (JRM): Every domain of continuity for **B** + GC, and every Church domain for **CRM**, has an almost negative definition.

Extended bar induction EBI_0 is monotone bar induction over $A^{\mathbb{N}}$.
Renardel de Lavalette proved that **EL** plus the restriction of EBI_0 to arithmetical A is arithmetically conservative over $\mathbf{IDB}_1$.
Conjecture (Troelstra): **EL** + EBI_0 has the same proof strength as the theory of finitely iterated positive inductive definitions.

A sensible suggestion

Anne Troelstra invented and used many nonclassical principles consistently extending accepted parts of intuitionistic mathematics. These include:

- Extended Church's Thesis ECT_0
- the Generalized Continuity principle GC
- the Uniformity Principle UP
- the principle of analytic data
- Troelstra's extension principle for lawless sequences

His name is usually attached only to the last of these, which is not the most useful for intuitionistic mathematics.

Maybe GC ought to be called "Troelstra's continuous choice principle" by analogy with Kleene's strong version of Brouwer's continuous choice principle, which it extends.

This talk has aimed to be a kind of botanical travelogue through Anne's books and papers, focused on a few genera and species, the mathematical analogues of the brambles he discovered in nature. Thank you for listening.